الاعداد العقدية
Complex Numbers

المحاضرة الاولى

الهدف من المحاضرة هو تعريف الطالب مايلي :

1. مفهوم الاعداد العقدية (Complex Numbers) .
2. نظم الاحداثيات التي تمثل العدد العقدي وصيغه .
3. تمثيل العدد العقدي (رياضيا) و (هندسيا).

مصطفى ابراهيم حميد

COMPLEX ANALYSIS FORMULA SHEET

Real number representation

Real number represent graphically in one dimension (either horizontal or vertical) as shown.

Due to *Quadratic* Algebric Equation ;

$$ax^2 + bx + C = 0$$

The solution with be x_1 and x_2 . The square root of ($\sqrt{b^2 - 4ac}$) may be (posative , negative or zero)

The *negative* value will be expressed as (*complex number*)

Complex Numbers represent by:

1. ***Rectangular*** **coordinate representation**

$$z = x + iy \implies \text{as a point with } (x\ ,y)$$

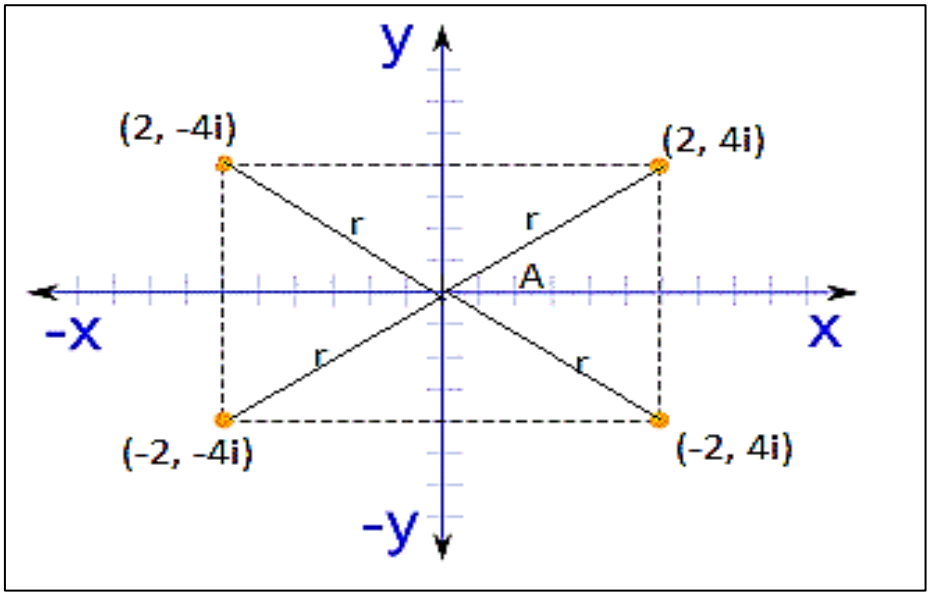
$\boldsymbol{x}$ *and* $\boldsymbol{y}$ are real numbers .

$\boldsymbol{x}$ are the real part of $z \implies x = Re(z)$

$\boldsymbol{y}$ are the imaginary part of $z \implies y = Im(z)$

$$z = Re(z) + Im(z).\boldsymbol{i}$$

Argand Plane **or** ***Complex Plane***



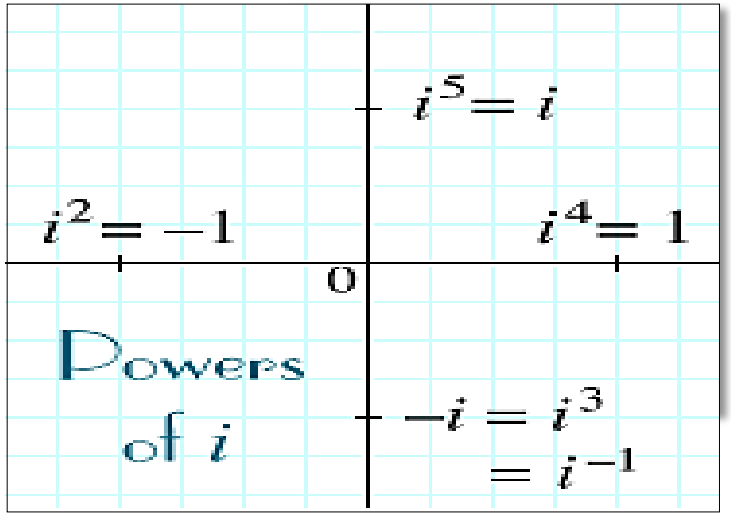
Complex unit

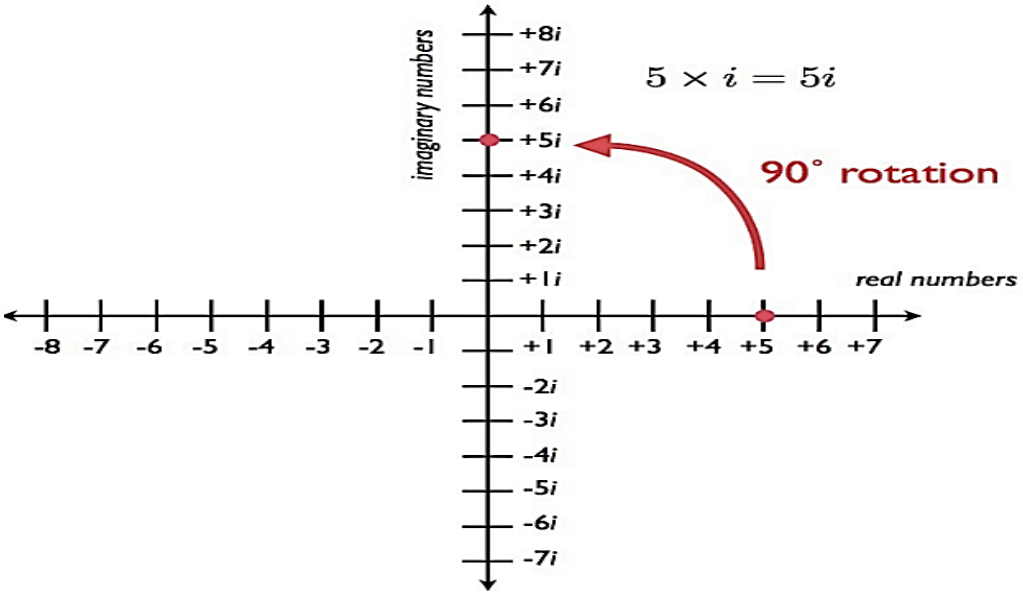
Now,

$$\boxed{\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \end{aligned}}$$

$$i^5 = i^2.i^2.i = (i^2)^2.i = (-1)^2.i = +i$$

$$i^{101} = (i^2)^{50}.i = (-1)^{50}.i = +i$$





2. *Polar* coordinate representation

$z = re^{i\theta}$ *specified by* $(\boldsymbol{r}, \boldsymbol{\theta})$

$z = r[cos\theta + isin\theta]$ *specified by* (r, θ) ***known as Euler's formula***

$z = r \angle \theta$

$$z = \boldsymbol{r} \angle \boldsymbol{\theta}$$

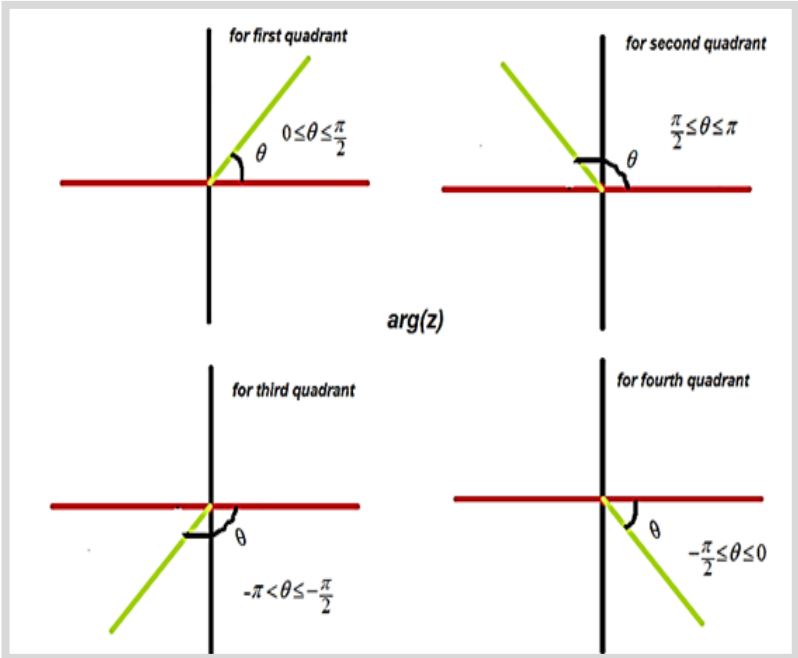
- **Length** of z
- **Amplitude** z
- **Modulus** z
- **Absolute value**

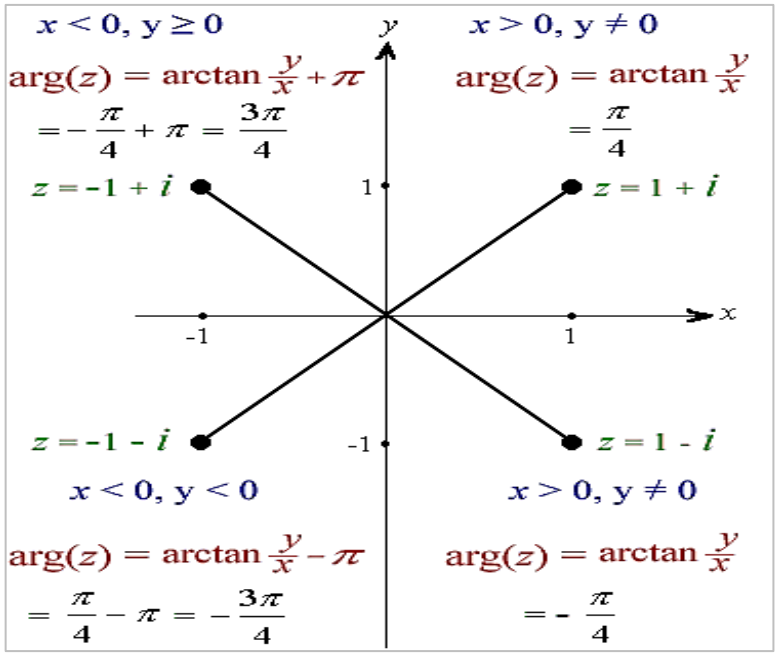
$$|z| = \sqrt{x^2 + y^2} = r$$

r is the " ***radius of circle*** " **centered** at orign.

θ is called '***angle*** " or , " ***argument***' or '***phase*** " represent the direction of Z and can be evaluated by :

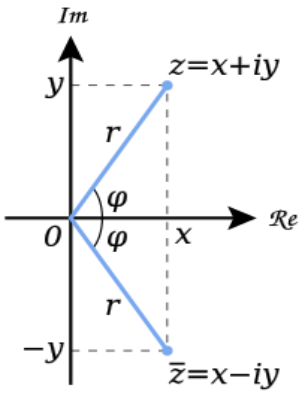
$$\theta = tan^{-1}(\frac{y}{x})$$

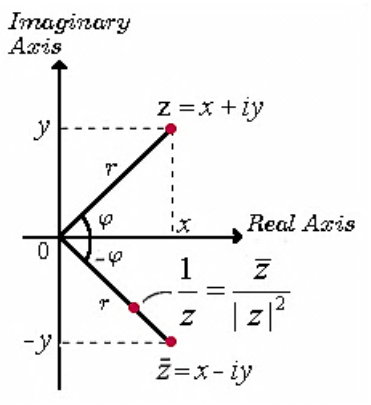


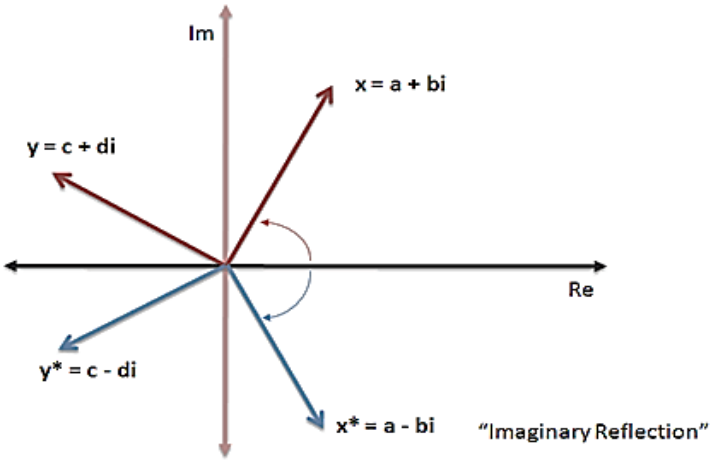


Complex conjugate of z Represented by $\bar{z}$ or z^*

$$\bar{z} = x - iy \implies \quad as\ a\ point\ with\ (x\ , -y)$$







Inflection of z

$$-z = -x - iy \implies \quad as\ a\ point\ with\ (-x\ ,-y)$$

Absolute value of z

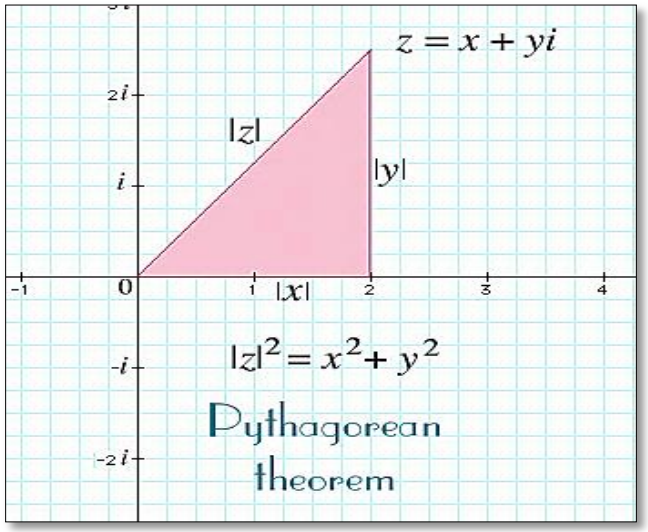
Also called , • **Length** of z

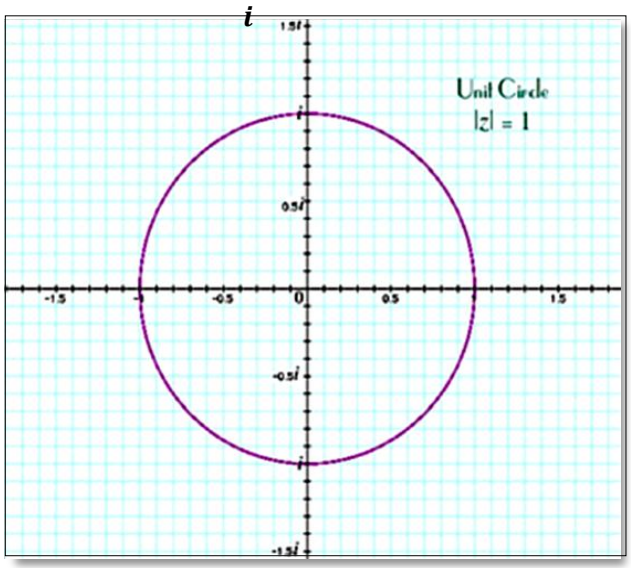
- **Amplitude** z
- **Modulus** z

$$|z| = \sqrt{x^2 + y^2} = r$$

r is the " ***radius of circle*** "

centered at $(x\ ,y)$.





Distance between z_1 *and* z_2

Represented by $|z_1 - z_2|$ and given by :

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Power of z

Or ,

$$z^n = r^n e^{(in\theta)}$$

$$(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$$

"*De Moivers theorem*"

$$z = x + iy = |z|(cos\,\theta + isin\theta) = |z|e^{i\theta}$$

PASCAL TRIANGLE

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b \ \ldots\ldots\text{etc}$$

Roots of z

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \qquad k=0,1,2,3,\ldots\ldots,(n-1)$$

$z_0, z_1, z_2 \ldots$ *the roots of* z

Complex Numbers

المحاضرة الثانيـــة

Examples

1. Verify

a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$

b) $\left(\frac{1+2i}{3-4i}\right)+\left(\frac{2-i}{i}\right)=\frac{-2}{5}$

$$\Rightarrow \frac{1+2i}{3-4i}*\frac{3+4i}{3+4i}+\frac{2-i}{5i}*\frac{-5i}{-5i}$$

$$=\frac{3+4i+6i-8}{25} \qquad =\frac{-10}{25}=\frac{-2}{5}$$

c) $\frac{5}{(1-i)(2-i)(3-i)}=\frac{1}{2}\,i$

$$\frac{5}{(1-i)(2-i)(3-i)}=\frac{5}{(1-3i)(3-i)}=\frac{5}{3-i-9i-3}=\frac{5}{-10i}=\frac{-i}{2i^2}$$

$$=+\frac{i}{2} \quad \text{since } i^2=-1$$

2. Simplify $\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)$

Solution:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)=\frac{1}{2+3-3i+2i}=\frac{1}{5-i}*\frac{5+i}{5+i}=\frac{5+i}{25-1}=\frac{5+i}{24}$$

$$=\frac{5}{24}+\frac{1}{24}i$$

Question : Express $X, Y, and\ |z|^2$ in terms of $Re(z)$ and $Im(z)$

Solution:

$$z=x+iy \quad \dots(1)$$

$$\underline{\bar{z}=x-iy} \quad \dots(2)$$

Add (1) and (2) ,

$$z + \bar{z} = 2x \Longrightarrow \boxed{X \equiv Re(z) = \frac{z + \bar{z}}{2}}$$

Now, subtract (1) and (2),

$$z = x + iy$$
$$\underline{\bar{z} = x - iy}$$

$$z - \bar{z} = 2yi \Longrightarrow \boxed{Y \equiv Im(z) = \frac{z - \bar{z}}{2i}}$$

$$|z|^2 = zz^* = \boxed{x^2 + y^2} = \left(Re(z)\right)^2 + \left(Im(z)\right)^2$$

Example: Simplify

$$\frac{-1 + 3i}{2 - i} * \frac{2 + i}{5} = -1 + i$$

Basic of algebraic properties of *z*, verify a few algebraic properties of *z*.

1. The commutative laws.

$$z_1 \pm z_2 = z_2 \pm z_1, \qquad z_1 z_2 = z_2 z_1$$

2. The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Now,

$$\frac{1}{z_1 z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right), \quad (z_1 \neq 0, z_2 \neq 0, z_1 z_2 \neq 0)$$

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, \quad \frac{z_1}{z\ z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right), \quad (z_3 \neq 0, z_4 \neq 0, z_3, z_4 \neq 0)$$

Complex Numbers

Absolute value of Z

With that interpretation in mind, we introduce the ***length*** , ***amplitude*** , ***absolute value*** or ***modulus*** of the complex number its the length when thinking of it as a vector :

If $z = x + iy$ then $|z| = \sqrt{X^2 + y^2}$

$|z|$ is the distance between the point (x, y) and the origin.

Question : what are the main difference between the absolute value of real numbers and complex numbers.

$$|z| = \sqrt{x^2 + y^2}$$

$|z_1 - z_2| =$ is the distance between the points representing the complex numbers z_1 a . When $y = 0$, $|z| = x$ usual absolute value in the real numbers system. from the point(x_2, y_2) to the point (x_1, y_1).

Example:

Compute the absolute value for each of the complex numbers : $z_1 = 1 + 2i$, $z_2 = -$

.

Solution:

$$z_1 = x_1 + iy_1 \Longrightarrow z_1 = 1 + 2i \Longrightarrow |z_1| = \sqrt{1 + 4} = \sqrt{5} z_2$$
$$|z_2| = \sqrt{9 + 1} = \sqrt{10}$$

$$|z_1| = \sqrt{5} \cong 2, \qquad |z_2| = \sqrt{10} \cong 3$$

The distance between the points representing the complex numbers $z_1$and z_2

$$\boxed{|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Complex Numbers

***Example*:** If $z_1 = 1 + i, z_2 = 2 - 3i$, find $|z_1 - z_2|$

Solution:

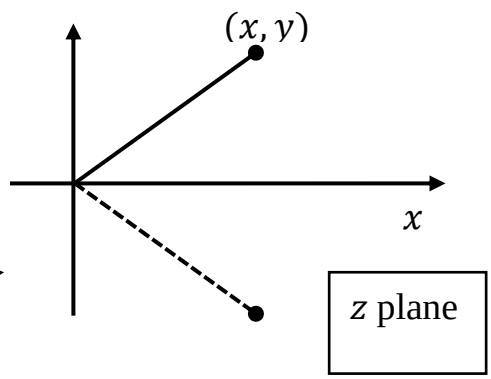
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(1-2)^2 + (1+3)^2} = \sqrt{1^2 + 4^2} = \sqrt{17}$$

Geometric Representation of z

1- Reflection on the real axis.

$$z = x + iy$$
$$\bar{z} = x - iy$$

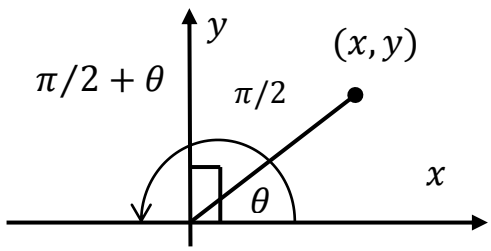
Reflection



2- Rotation of z

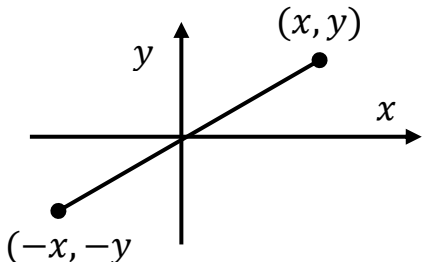
If z is non-zero then iz

$$z = x + iy$$
$$iz = ix - y$$
$$e^{\pi/2\, i} e^{i\theta} = e^{i(\theta + \pi/2)}$$



3- Inflection of z

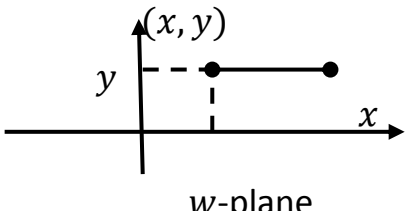
$$z = x + iy$$
$$-z = -x - iy$$



Operations

4- Translation of z

$$z = x + iy$$
$$w = 1 + z = 1 + x + iy$$



Operations in a comlpex plane

Complex Numbers

2022

المحاضرة الثالثة :

(1) رسم الاعداد العقدية ، جمعها طرحها بالرسم

(2) تطبيقات القيمة المطلقة للاعداد العقدية

Cartesian Coordinates

$$z = x + iy$$

It is natural to associate the complex number $z = x + iy$ with a point in the plane whose carteian coordinates are x and y.

a) The number $2 + i$ isrepresented by the point (2,1).

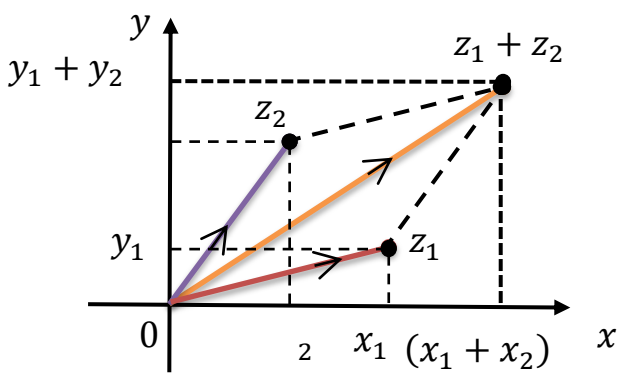
b) The number z can also be thought of as the directed line segment, or vector, from the origin to the point (x, y).

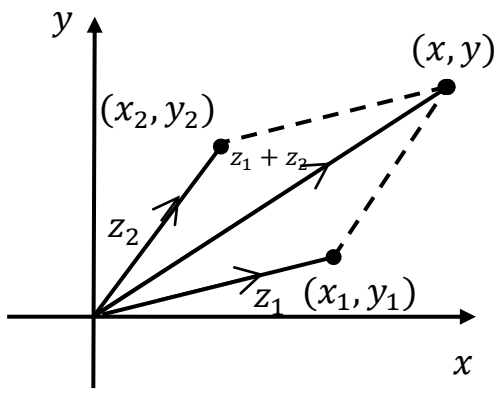
***Question*:** How can one represent the complex number?

Complex number may be represented either by a point in the plane, or as a vector starting from 0 to (x, y).

a) It also corresponds to a vector with those coordinates as its components.

Question **:** Represent vectorially $z_1 + z_2$





***NOTE*:** The addition of $(z_1 + z_1)$ is geometrically represented by the parallelogram law.

Exercises

1. Locate the numbers $z_1 + z_1$ and $z_1 - z_2$ vectorially, when:

a) $z_1 = 2i,\ \ z_2 = \frac{2}{3} - i$

b) $z_1 = (-\sqrt{3}, 1),\ z_2 = (\sqrt{3}, 0)$

c) $z_1 = (-3,1),\ z_2 = (1,4)$

d) $z_1 = x_1 + iy_1,\ z_2 = x_1 - iy_1$

Solution

a) $z_1 = 2i$ $\qquad$ $z_1 = 2i$

$z_1 + z_2 = \frac{2}{3} + i \qquad z_1 - z_2 = -\frac{2}{3} + 3i$

b) $z_1 = (-\sqrt{3}, 1),\ z_2 = (\sqrt{3}, 0)$

$z_1 = -\sqrt{3} + i \qquad z_1 = -\sqrt{3} + i$

$z_2 = \sqrt{3} \qquad \mp z_2 = \mp\sqrt{3} + 0i$

$z_1 + z_2 = \qquad z_1 - z_2 = -2\sqrt{3} + i$

c) $z_1 = (-3,1),\ z_2 = (1,4)$

$z_2 = 1 + 4i \qquad \mp z_2 = \mp 1 \mp 4i$

$z_1 + z_2 = -2 + 5i \qquad z_1 - z_2 = -4 - 3i$

d)

$$\begin{array}{ll} z_1 = x_1 + iy_1 & z_1 = x_1 + iy_1 \\ z_2 = x_1 - iy_1 & \mp z_2 = \mp x_1 \pm iy_1 \\ \hline z_1 + z_2 = 2x_1 & z_1 - z_2 = 2iy_1 \end{array}$$

Question: Show that the complex numbers corresponding to the points on the circle with center (0,1) and radius 3, satisfy the equation $|z - \blacksquare| = 3$ and conversely. **Or/** ***we refer to the set of points simply as the circle $|z - i| = 3$***

$$z = x + iy$$

$$|z - i| = |x + iy - i| = |x + i(y - 1)| = 3$$

$$= \sqrt{x^2 + (y-1)^2} = 3 \Longrightarrow \text{Circle of radius 3 centered at } (\mathbf{0}, \mathbf{1})$$

Question : Plot $|z - i| = 3$, if $z = 2 + 3i$,

$$|2 + 3i - i| = 3$$

$$|2 + 2i| = 3 \Longrightarrow \sqrt{2^2 + 2^2} = 4$$

$$x = 2, y = 2, r = 3$$

Absolute Value

- $|z_1 z_2| = |z_1||z_2|$,
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z}$

Conjugate of complex numbers :

- sum of the conjugate $\overline{z_1 + z_2} = \bar{z} + \bar{z}$
- Subtraction of the conjugate $\overline{z_1 - z_2} = \bar{z} - \bar{z}$
- Multiplication of the conjugate $\overline{z_1 z_2} = \bar{z}\,\bar{z}$
- Division of the conjugate $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{\ }}$

2. Show that

a) $\overline{\overline{z}+3i} = z - 3i$

b) $\overline{iz} = -i\,\overline{z}$

c) $\overline{\frac{(2+i)^2}{3-4i}} = 1$

d) $\left|\frac{2\overline{z}+5}{\sqrt{2}-i}\right| = \sqrt{3}|2z+5|$

Solution:

a) $\overline{\overline{z}+3i} = z - 3i$, or since $\boxed{\overline{\overline{\ }} = z}$

$$\overline{\overline{(x+iy)}+3i} = \overline{(x-iy)+3i} = \overline{(x-iy)} + \overline{3i} = (x+iy) - 3i = z - 3i$$

b) $\overline{iz} = \overline{i}\,\overline{z} = -i\,\overline{z}$

c) $\overline{\frac{(2+i)^2}{3-4i}} = 1$

$$\frac{\overline{(2+i)^2}}{\overline{3-4i}} = \frac{\overline{3+4i}}{3-4i} = \frac{3-4i}{3-4i} = 1$$

d) $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}|2z+5|$, since $\boxed{|z_1 z_2| = |z_1||z_2|}$

LH.S

$\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = |(2\overline{z}+5)|\left|(\sqrt{2}-i)\right|$

$|\sqrt{2}-i| = \sqrt{(\sqrt{2})^2 + (-1)^2} = \sqrt{2+1} = \sqrt{3}$

$|(2\overline{z}+5)| = \left|\left(2\overline{(x+iy)}+5\right)\right| = |(2(x-iy)+5)| = |(2x-2iy+5)| =$

$\left|((2x+5)-2iy)\right| = \sqrt{(2x+5)^2+(2y)^2}$

$$= 4x^2 + 20x + 25 + 4y^2$$

$\boxed{|z| = \sqrt{x^2+y^2}}$

R.H.S

$$\sqrt{3}|2z+5| = \sqrt{3}|(2x+2iy+5)| = \sqrt{3}|(2x+5)+2iy|$$

$$= \sqrt{3}\sqrt{(2x+5)^2+(2y)^2}$$

$$L.H.S \equiv R.H.S$$

Complex Numbers

2022

المحاضرة الرابعة :

1. الاعداد العقدية بالنظام القطبي .. أمثلة تطبيقية

Polar Form or Trigonometric Form of a Complex Number

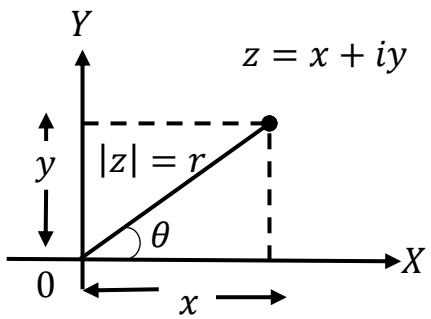
A complex number z, when written as

$$z = x + iy \quad or\,, \qquad z = x + jy$$

is said to be expressed in *rectangular form* ,also known as Cartesian coordinates .

But is also may be expressed in *polar form*

$$z = r\angle\,\theta$$



Geometrically, $|z|$ is the distance of the point from the origin r or can be represent by (A) .The number r (or A) is the length (or amplitude, modulus of the vector representing z; that is,

$$\boxed{r = |z|} = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \quad (r \geq 0)\ ,\ \theta = tan^{-1}(\tfrac{y}{x})$$

Conjugate of a Complex Number

The conjugate z^* or $\bar{z}$ of a complex number z is obtained by ***changing the sign of the imaginary part of the number.***

Conjugate of $z = z^* = Re\{z\} - iIm\{z\} = r\angle - \theta$

Now mathematical operation of complex number :

1. Addition $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
3. Multiplication $z_1 z_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$
4. Division $\frac{z_1}{z}$ —

Complex Numbers

2022

5. Reciprocal $\frac{1}{z} = \frac{1}{r}\angle -\theta$
6. Square root $\sqrt{z} = \sqrt{r}\angle\ {}^{\theta}/_{2}$
7. Complex conjugate $z^* = x - jy = r\angle -\theta$

$$+90^o = +\sqrt{-1} = +j = 1\angle +90^o = 0 + j1$$

$$-90^o = -\sqrt{-1} = -j = 1\angle -90^o = 0 - j1$$

$$\mp 180^o = (\ \sqrt{-1}\ )^2 = -1 = 1\angle \mp 180^o = -1 + j\,0$$

Converting Polar Form into Rectangular Form, (P→R)

(From **P**olar form to **R**ectangular form)

$$6\angle 30^o = x + jy$$

However,

$$x = Acos\theta \quad , y = Asin\theta$$

Therefore,

$$6\angle 30^o = (6cos\theta) + j(6sin\theta)$$

$$=(6cos\theta) + j(6sin\theta)$$

$$=(6cos30^o) + j(6sin30^o)$$

$$=(6 \times 0.866) + j(6 \times 0.5)$$

Converting Rectangular Form into Polar Form (R→P)

(From **R**ectangular form to **P**olar form)

$$(\mathbf{5.2} + \boldsymbol{j}\mathbf{3}) = \boldsymbol{A}\angle\boldsymbol{\theta}$$

Where $\boldsymbol{A} = \sqrt{\mathbf{5.2^2 + 3^2}} = \mathbf{6}$ and $\boldsymbol{\theta} = \boldsymbol{tan}^{-1}\frac{\mathbf{3}}{\mathbf{5.2}} = \mathbf{30}^{\boldsymbol{o}}$

Hence , $(\mathbf{5.2} + \boldsymbol{j}\mathbf{3}) = \mathbf{6}\angle\mathbf{30}^{\boldsymbol{o}}$

Complex Numbers

2022

Multiplication in Polar Form

$$\boldsymbol{Z_1 Z_2 = (A_1 A_2)\angle(\theta_1 + \theta_2)}$$

Multiplying together 6 $\angle 30^o$ and 8 $\angle -45^o$ in polar form gives us:

$$\boldsymbol{Z_1 Z_2 = (6 \times 8)\angle(30^o + (-45^o)) = 48\angle -15^o}$$

Division in Polar Form

$$\boldsymbol{\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right)\angle\theta_1 - \theta_2}$$

$$\boldsymbol{\frac{Z_1}{Z_2} = \left(\frac{6}{8}\right)\angle 30^o - (-45^o) = 0.75\angle 75^o}$$

Polar Coordinates

Let r and θ be polar coordinates of the point (x, y) corresponding to a nonzero complex number $z = x + iy$

Since, $x = r\cos\theta$, $y = r\sin\theta$

$$\boxed{z = r(\cos\theta + i\sin\theta)} \quad , \qquad \textit{polar form of number } z$$

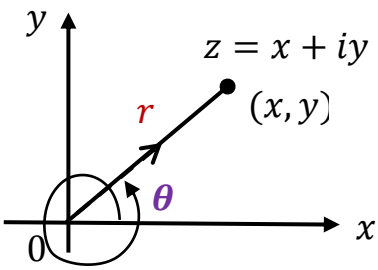
For example,

$$1 + i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$$

The number θ is called an *argument* of z, (also called, '*phase*' *and* '*angle*')

θ is the directed angle measured from the positive x-axis to a point p and called (*argument* of z) and is denoted by : arg ; $\theta = \arg z_1$

Hence the angles will be measured in *radian* and positioned in the *counterclockwise sense*. $\boxed{\tan\theta = \frac{y}{x}}$



Complex Numbers

2022

Important identity

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

Question : Find the value of (iz)

$$iz = \underbrace{(1)\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]}_{\boxed{i}}\underbrace{r(\cos\theta + i\sin\theta)}_{\boxed{z}}$$

$$iz = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

iz has an important applications in Optics i.e. (*circular polarization*).

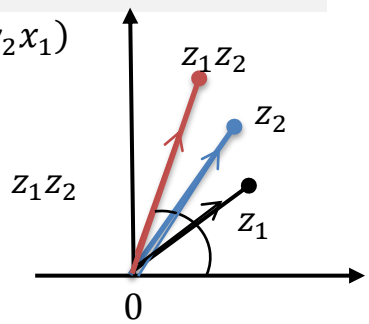
Example:

$\theta_1 = \arg(1+i)$, $\theta_1 = arg(-1-i) \Longrightarrow \tan\theta_1 = \tan\theta_2 = 1$....**PLOT!!**

Question : Use Cartesian coordinate system representation of z , simplified the multiplication of two different complex numbers.

$$z_1 z_2 = (x_1 + iy_2)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1)$$

$$(\mathbf{1} + \boldsymbol{i})(\mathbf{2} + \mathbf{3}\boldsymbol{i}) = \mathbf{2} + \mathbf{5}\boldsymbol{i} - \mathbf{3} = -\mathbf{1} + \mathbf{5}\boldsymbol{i}$$

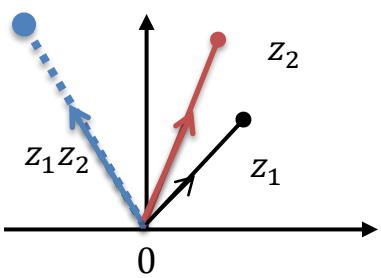


$z_1 = (1+i) \rightarrow x_1 = 1, y_1 = 1,\ r_1 = \sqrt{2},\ \theta_1 = 45°$,

$\tan^{-1}\frac{3}{2}$ or $\theta_2 = 56.31°$,

For the ***final results*** $-1 + 5i$, will be :

$r = 5.1$, $\theta = \tan^{-1} -\frac{5}{1}$, $\theta = 101.31°$



Complex Numbers

2022

المحاضرة الخامسة :

1. الدوال الاسية العقدية وخواصها مع امثلة .
2. قوى الاعداد العقدية مع امثلة .
3. جذور العدد العقدي مع أمثلة .

Exponential Function الدالة الاسية العقدية

The polar form is $z = r(\cos\theta + i\sin\theta)$,

or , $z = re^{i\theta}$

Where $\boxed{e^{i\theta} = (\cos\theta + i\sin\theta)}$, is ***Euler's formula*** , and $e^{i\theta}$(*complex exponential function*)

NOTE:

$$(1)\ z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$(2)\ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Question **: Relate trigonometric function with complex exponential function**

الربط بين الدوال الاسية العقدية والدوال المثلثية الحقيقة

Proof: $\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \dots(1)$$

$$\underline{e^{-i\theta} = \cos\theta - i\sin\theta} \quad \dots(2)$$

Adding (1) and (2) : $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Longrightarrow \boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$

subtracting (1) and (2)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Complex Numbers

2022

Powers of complex numbers

$\boxed{z^n = r^n e^{in\theta}}$, $(n = 0, \pm 1, \pm 2, \dots)$, **where** $\left(e^{i\theta}\right)^n = e^{in\theta}$

Or $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = \cos\phi + i\sin\phi}$,

$(n = 0, \pm 1, \pm 2, \dots)$ $(\boldsymbol{de\ Moivre's\ theorem}!!)$

Where $\boxed{\phi = \boldsymbol{n}\theta}$

Roots of complex number z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \qquad \boxed{k = 0,1,2,\dots,n-1}$$

Example

1. Solve the equation $z^3 + 1 = 0$, or in other words
2. Find z for the equation $z = (-1)^{\frac{1}{3}}$
3. Find the third roots(***ONLY***) of the equation $z^3 + 1 = 0$

Solution

$$z^3 = -1, = (-1)^{\frac{1}{3}},$$

$$x = -1, y = 0, r = \sqrt{x^2 + y^2} \Longrightarrow r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Longrightarrow \quad \theta = \pi, \qquad n = 3, \qquad k = 0,1,2$$

1st *root* $\boxed{k = 0}$, $z_1 = (1)\left[cos\frac{\pi + 2(0)\pi}{3} + i\,sin\frac{\pi + 2(0)\pi}{3}\right]$

$$= \frac{1}{2} + i0.866 = 0.5 + i0.866$$

2nd *root* $\boxed{k = 1}$,

$$z_2 = \left[cos\frac{\pi + 2\pi}{3} + i\,sin\frac{\pi + 2\pi}{3}\right]$$

$$= \cos\pi + i\sin\pi = -1 + 0i$$

3rd *root* $\boxed{k = 2}$,

$$z_3 = \left[cos\frac{\pi + 4\pi}{3} + i\,sin\frac{\pi + 4\pi}{3}\right]$$

$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = 0.5 - 0.866i$

The 3rd root $k = 2$ was $(0.5\ , -0.866)$

Examples

1. Find one value of $\arg(z)$ when

a) $z = \dfrac{-2}{1+i\sqrt{3}}$, **b)** $z = \dfrac{i}{-2-2i}$, **c)** $z = \left(\sqrt{3}-i\right)^6$

Solution:

a) $z = \dfrac{-2}{1+i\sqrt{3}}$, $\dfrac{-2}{1+i\sqrt{3}}\ \dfrac{1-i\sqrt{3}}{1-i\sqrt{3}} = \dfrac{(-2)(1-\sqrt{3}i)}{1+3}$

$= \frac{-2+2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

$\therefore \boldsymbol{x} = -\dfrac{\mathbf{1}}{\mathbf{2}} = \mathbf{-0.5}$

$\boldsymbol{y} = \dfrac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{3}} = \mathbf{0.866} \Longrightarrow \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\dfrac{1/2\ \sqrt{3}}{/}\right)$

$\theta = \tan^{-1}\ (-\sqrt{3}) \Longrightarrow \boxed{\theta = -60}, \theta = 180° - 60° = 120°$

b) $\dfrac{i}{-z-2i}$,

$\dfrac{i}{-z-2i} = \dfrac{i}{-2-2i} * \dfrac{-2+2i}{-2+2i} = \dfrac{i(-2+2i)}{\qquad\qquad} \qquad 8$

$\dfrac{i}{-z-2i} = \dfrac{-2i+2}{8}$

$\dfrac{i}{-z-2i} = \dfrac{1-i}{4} = \dfrac{\mathbf{1}}{\mathbf{4}} - \dfrac{\mathbf{1}}{\mathbf{4}}i$

$x = \frac{1}{4}, y = -\frac{1}{4} \Longrightarrow \theta = \tan^{-1}\left(\frac{-1/4}{/}\right) = \tan^{-1}(-1) = -45° = 315° = -\frac{\pi}{4} \equiv \frac{3\pi}{4}$

c) $z = \left(\sqrt{3}-i\right)^6$

$x = \sqrt{3}$

$y = -1, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$

2022

$$n\theta = 6 * \theta = -30° * 6 = -180° = 360° - 180° = \pi$$

2. Use the polar form to show that

a) $i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + i2\sqrt{3}$;

b) $5i(2 + i) = 1 + 2i$;

c) $(-1 + i)^7 = -8(1 + i)$;

d) $(1 + i\sqrt{3})^{-10} = 2^{-11}(-1 + i\sqrt{3})$.

Solution

a) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$x = \sqrt{3}$

$y = +1$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$L.H.S \quad (\sqrt{3} + i)(\sqrt{3} + i) = 3 - 1 + 2\sqrt{3}i = 2 + 2\sqrt{3}i$

b) $5i/(2 + i)$

$$\frac{5i}{2+i} = \frac{5i}{2+i} * \frac{2-i}{4+1} = (\frac{5(1+2i)}{5} = 1 + 2i$$

c) $(-1 + i)^7 = -8(1 + i)$

<u>L.H.S</u> $(-1 + i)^7$

$x = -1$

$y = 1, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \theta = \tan^{-1}(-1) \Rightarrow, \theta = -45°,$

Then $\theta = 180° - 45° = 135°$, Or, $\theta = \frac{3\pi}{4}$

$$7\theta = 7 * \frac{3\pi}{4} = \frac{21}{4}\pi$$

$$e^{i.\frac{21}{4}\pi} \stackrel{?}{=} e^{.\frac{\pi}{4}i}$$

<u>R.H.S</u> $-8 - 8i \Rightarrow x = -8$

$y = -8, \quad \theta = \tan^{-1}(1) \Rightarrow \theta = \frac{\pi}{4}$

$$\theta = 45° + 180° = 225°$$

Or, $\theta = \frac{\pi}{4} + \pi$, $\theta = 5\frac{\pi}{4}$

2022

Roots: $i, (1 \pm \sqrt{3})/2$

d) $8^{\frac{1}{8}}$

$z = 8^{\frac{1}{6}} \Longrightarrow z^6 - 8 = 0, \boxed{8^{\frac{1}{6}} = 0.1666}$ using scientific calculator

$\left.\begin{array}{l} x = 8 \\ y = 0 \end{array}\right\} \Longrightarrow r = 8 \; and \; \boxed{\theta = 0}$

$n = 6, \quad \boxed{k = 0,1,2,3,4,5,6,7}$

$$w_0 = z^{\frac{1}{6}} = 1\left[\cos\left(\frac{0 + 2(0)\pi}{6}\right) + \mathrm{i}\sin\left(\frac{0 + 2(0)\pi}{6}\right)\right] = 1.414[1 + 0i] \qquad w_0 = 1.414$$

$$w_1 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0 + 2\pi}{6}\right)}_{= 60^\circ} + \mathrm{i}\sin\left(\frac{0 + 2\pi}{6}\right)\right] = 1.414[0.5 + 0.866i],$$

$$w_1 = 0.707 + 1.225i$$

$$w_2 = z^{\frac{1}{6}} = 1\left[\underbrace{\left(\frac{0 + 4\pi}{6}\right)}_{=120^\circ} + i\,sin\left(\frac{0 + 4\pi}{6}\right)\right] = 1.414[-0.5 + 0.866i],$$

$$w_2 = -0.707 + 1.225i$$

$$w_3 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0 + 6\pi}{6}\right)}_{=\pi} + i\,sin\left(\frac{0 + 6\pi}{6}\right)\right] = 1.414[-1 + 0i], \qquad w_3 = -1.414$$

$$w_4 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0 + 8\pi}{6}\right)}_{/\ *180=240^\circ} + i\,sin\left(\frac{0 + 8\pi}{6}\right)\right] = 1.414[-0.5 - 0.866i], \qquad w_4 = -0.707 - 1.225i$$

$$w_5 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0 + 10\pi}{6}\right)}_{\frac{5}{3}\pi=300^\circ} + i\,sin\left(\frac{0 + 10\pi}{6}\right)\right] = 1.414[0.5 - 0.866i], \qquad w_5 = 0.707 - 1.225i$$

$$w_6 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0+12\pi}{6}\right)}_{2\pi=360^\circ} + i\,sin\left(\frac{0+12\pi}{6}\right)\right] = 1.414[1 + 0i],$$

$$w_6 = 1.414$$

$$w_7 = z^{\frac{1}{6}} = 1\left[\cos\underbrace{\left(\frac{0 + 14\pi}{6}\right)}_{=420^\circ} + \mathrm{i}\sin\left(\frac{0 + 14\pi}{6}\right)\right]$$

$$= 1.414[0.5 + 0.866i],$$

Roots are then,

$$\pm\sqrt{2}, \frac{(1 \pm i\sqrt{3})}{\sqrt{2}}, \frac{(-1 \pm i\sqrt{3})}{\sqrt{2}}$$

Examples

4. Find a value of $\arg z$ when $z_1 \neq 0$ and $z_2 \neq 0$;

a) $z = \frac{z_1}{z_2}$ **b)** $z = z^n (n = 1,2, \dots)$ **c)** $z = z^{-1}$

Solution *:*

a) $Z = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta}}{{}_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$

Since, $\boxed{\arg(z) = \arg(z_1) - \arg(z_2)}$

b) $z = z^n (n = 1,2, \dots)$

$z = r^n e^{in\theta} = r^n[\cos(n\theta) + \sin(n\theta)] \Longrightarrow \arg z^n \Longrightarrow n \arg z_1$

c) $z = z^{-1} \Longrightarrow r^{-1} e^{-i\theta_1}, \quad \arg z^{-1} = -\arg z_1$

5. Find four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficient.

$$z^4 + 4 = 0 \Longrightarrow (z^2 + 2i)(z^2 - 2i) = 0$$

$$z^2 + 2i = 0 \Longrightarrow z^2 = -2i, \qquad z = (-2i)^{\frac{1}{2}}$$

$$or, \; z^2 - 2i = 0 \Longrightarrow z^2 = 2i, \; z = (2i)^{\frac{1}{2}}$$

$$\left.\begin{array}{l} z = (-2i)^{\frac{1}{2}} \\ z = (2i)^{\frac{1}{2}} \end{array}\right\}$$

Compute the solution!

<u>Roots</u>

Or, $z^4 + 4 = 0$

$$z^4 + 2 + 2z + 2 - 2z = 0$$
$$z^4 + 2z + 2 + 0 - 2z + 2 = 0$$
$$z^4 + 2z + z^2 + 2 - z^2 + 0 - 2z + 2 = 0$$

2022

$(z^2+2z+2)(z^2-2z+2)=0$

(z^2+2z+2) or $(z^2-2z+2)=0$ *then complete the solution*.

Example:

Let $z_1=2+3i$ and $z_2=4+5i$, Find (in the form $x+iy$):

a) z_1z_2
b) $(z_1+z_2)^2$,
c) $1/z_1$,
d) z_1/z_2.

Solution:

$$z_1z_2=(2+3i)(4-5i)=8+2i+15=23+2i$$

$$(z_1z_2)^2=(23+2i)^2=23^2+2(23)(2)i-4=\cdots$$

$z_1=2+3i$
$z_2=4-5i$

$z_1+z_2=6-2i$

$(z_1+z_2)^2=(6-2i)^2=36-24i-4=32-24i$

$$\frac{1}{2+3i}*\frac{2-3i}{2-3i}=\frac{2-3i}{4+9}=\frac{2-3i}{16}=\frac{1}{8}-\frac{3}{16}i$$

$$\frac{z_1}{4-5i}*\frac{4+5i}{4+5i}=\frac{(2+3i)(4+5i)}{16+25}=\frac{-7+22i}{41}$$
$$=-\frac{7}{41}+\frac{22}{41}i$$

Complex Numbers

2022

المحاضرة السادسة :

التطبيقات الفيزيائية للاعداد والدوال العقدية .

Physical Applications

1. Wave Equation

The mathematical representation of wave is given by $y = Ae^{i(kx-wt)}$, Find the intensity I .

Solution:

$$\boxed{z = re^{i\theta}} \Longrightarrow \boxed{\begin{array}{c} z \equiv y \\ r \equiv A = \text{amplitude} \\ \theta \equiv angle\ (\ or\ phase.) \equiv (kx - wt) \\ In\ PHYSICS \\ \theta\ \ is\ called\ (\boldsymbol{PHASE}\ ) \end{array}}$$

$$since,\ \boxed{zz^* = |z|^2 = r^2}\ \ in\ \boldsymbol{mathematics}$$

$$\boxed{yy^* \equiv y^- = |y|^2 = A^2}\ \ in\ \boldsymbol{physics}$$

i.e. $yy^* = |y|^2 = Ae^{\boldsymbol{i}(kx-wt)} * Ae^{-\boldsymbol{i}(kx-wt)}$

Since ,

$$Ae^{\boldsymbol{i}(kx-\omega t)} * Ae^{-\boldsymbol{i}(kx-wt)} = \boldsymbol{A^2 e^{i(kx-\omega t)-i(kx-\omega t)}} = \boldsymbol{A^2 e^0}$$

$$|y|^2 = \boldsymbol{A^2} * (\boldsymbol{1}) = \boldsymbol{A^2}$$

$$\Longrightarrow |y|^2 \equiv \quad \boldsymbol{I = A^2}$$

Activity : Find $y_1 + y_2, y_1 - y_2, y_1 y_2, |y_1 - y_2|$ for wave of the same phase of different amplitude.

Answer **:**

$\boldsymbol{y_1 + y_2}$

$$y \ = A_1 e^{\boldsymbol{i}(k_1 x_1 - \omega_1 t_1)}$$

$$y \ = A_2 e^{\boldsymbol{i}(k_2 x_2 - \omega_2 t_2)}$$

$$y_1 + y_2 = A_1 \quad {}^{(k_1 x_1 - \omega_1 t_1)} + A_2 e^{\boldsymbol{i}(k_2 x_2 - \omega_2 t_2)}$$

for wave of the same phase of different amplitude ,

$$(k_1 x_1 - \omega_1 t_1) \equiv (k_2 x_2 - \omega_2 t_2) = (kx - \omega t)$$

Then ,

$$y_1 = A_1 e^{\boldsymbol{i}(kx - \omega t)}$$

$$y_2 = A_2 e^{\boldsymbol{i}(kx - \omega t)}$$

$$y_1 + y_2 = (A_1 + A_2)\ [cos(kx - \omega t) + \boldsymbol{i}\ sin\ (kx - \omega t)]$$

$$y_1 + y_2 = (A_1 + A_2)\ [cos(kx - \omega t) + \boldsymbol{i}\ sin\ (kx - \omega t)]$$

$$|y_1 + y_2| = (A_1 + A_2)$$

For, $\boldsymbol{y_1 - y_2}$

$$y_1 - y_2 = (A_1 - A_2)\ [cos\left(kx - \omega t\right) + \boldsymbol{i} sin\ \left(kx - \omega t\right)]$$

Now,

$\boldsymbol{|y_1 - y_2|}$

$$|y_1 - y_2| = \sqrt{(A_1 - A_2)^2 cos^2(kx - \omega t) + (A_1 - A_2)^2 sin^2(kx - \omega t)}$$

$$|y_1 - y_2| = \sqrt{(A_1 - A_2)^2 \{cos^2(kx - \omega t) + sin^2(kx - \omega t)\}}$$

Then ,

then ,*Capacitive* ,

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(50)(20)} F \quad , \quad C = \frac{10^6}{2\pi(50)(20)} \mu F = 159.2\ \mu F$$

(d) *Impedence* , Z=$15\angle -60^o$

$$Z=15\angle -60^o = 15[\cos(60^o) + j\sin(-60^o)] = 7.50 - j12.99\Omega.$$

Hence,
resistance =7.50Ω and capacitivie reactance , $X_C = 12.99\Omega$.
Since

$$\frac{1}{2\pi f C} \qquad \text{then capacitive,} \quad C = \frac{1}{2\pi f X_C} = \frac{10^6}{2\pi(50)(12.99)} \mu\text{F} = 245\mu\text{F}$$

Example :

An alternative voltage of 240V, 50Hz is connected across an impedance of $(60 - j100)\Omega$.

Determine the :

(a) resistance
(b) capacitance
(c) magnitude of the impedance and its phase angle and
(d) current flowing

Solution

Impedance $Z = (60 - j100)\Omega$. Hence resistance = 60Ω.

(a) Capacitive reactance $X_C = 100\Omega$, and Since

$$X_C = \frac{1}{2\pi f C}$$

then *capacitance* ,

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(50)(100)} = \frac{10^6}{2\pi(50)(100)} \mu\text{F} = 31.83\ \mu\text{F}$$

(c) *Magnitude of impedance* ,

$$|Z| = \sqrt{(60)^2 + (-100)^2} = 116.6\ \mu\text{F}$$

Phase angle, or $\arg Z = tan^{-1}\left(\frac{-100}{60}\right) = -59.04^o$

(b) *Current flowing* ,

$$I = \frac{V}{Z} = \frac{240\angle 0^o}{116.6\angle -59.04^o} = 2.058 \angle 59.04^o \text{ A}$$

Example :

For the parallel circuit shown in Fig., determine (,using complex numbers)the values of:

(a) current I ,and
(b) its phase relative to the 240V supply

Solution

- ✓ Current $I = \frac{V}{Z}$.
- ✓ Impedance Z for the three-branch parallel circuit is given by :

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

Where

$$Z_1 = 4 + j3\,, \quad Z_2 = 10 \text{ and } Z_3 = 12 - j5$$

- ✓ Admittance , $Y_1 = \frac{1}{Z_1} = \frac{1}{4+j3}$

$$= \frac{1}{4+j3} \times \frac{\mathbf{4-j3}}{\mathbf{4-j3}} = \frac{4-j3}{4^2+3^2}$$

$$Y_1 = 0.160 - j0.120 \text{ siemens}$$

- ✓ Admittance , $Y_2 = \frac{1}{Z_2} = \frac{1}{10}$

$$Y_2 = 0.10 \text{ siemens}$$

- ✓ Admittance , $Y_3 = \frac{1}{3} = \frac{1}{12-j5}$

$$= \frac{1}{12-j5} \times \frac{\mathbf{12+j5}}{\mathbf{12+j5}} = \frac{12+j5}{12^2+5^2}$$

$$Y_3 = 0.0710 + j0.0296 \text{ siemens}$$

Total admittance, $\boldsymbol{Y} = \boldsymbol{Y_1} + \boldsymbol{Y_2} + \boldsymbol{Y_3}$

$$= (0.160 - j0.120) + (0.10) + (0.0710 + j0.0296)$$
$$= 0.331 - j0.0904$$
$$\boldsymbol{Y} = 0.343\angle - 15.28^o \text{ siemens}$$

Current , $I = \frac{V}{Z} = VY$

$$I = (240\angle 0^o)\,(0.343\angle - 15.28^o) = 82.32\angle - 15.28^o\ A$$

$$\boldsymbol{I = 82.32\angle - 15.28^o\ A}$$

3.Mechanics

(a) ***Three coplanar forces***

Determine the

- ✓ magnitude and
- ✓ direction of the resultant of the three coplanar forces given below ,when they act at a point :
  - Force A, 10N acting at 45° from the poisitive horizontal axis ,
  - Force B , 8N acting at 120° from the positive horizontal axis ,
  - Force C , 15N acting at 210° from the positive horizontal axis.

Solution

The space daigram is shown in the figure .

The forces may be written as complex numbers.
Thus

force A , $F_A = 10\angle 210^o$,

force B , $F_B = 8\angle 120^o$,

force C , $F_C = 15\angle 210^o$.

- The resultant force $= f_A + f_B + f_C$

$$= 10\angle 45^o + 8\angle 120^o + 15\angle 210^o$$

$$\boldsymbol{F} = 10(cos45^o + jsin\,45^o) + 8(cos120^o + jsin\,120^o) + 15(cos210^o + jsin\,210^o)\text{F}$$

$$= 7.071 + j7.071) + (-4.00 + j6.928) + (-12.99 - j7.5) = -9.919 + j6.499$$

$$\boldsymbol{F} = -9.919 + j6.499$$

Since

$-9.919 + j6.499$ lies in the second quadrant

- *Magnitude of resultant force*

$$F = \sqrt{(-9.919)^2 + 6.499^2} = 11.86N$$

- *Direction of resultant force*

$$= tan^{-1}\left(\frac{6.499}{-9.919}\right) = 146.77^o$$

since $-9.919 + j6.499$) lies in the second quadrant).

Solving the harmonic oscillator

A harmonic oscillator is governed by the equation

$$ma = -kx \qquad ....(1)$$

Where a is acceleration .

This provides us with the differential equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad ...(1)$$

We know that the solution of this equation can be written as sine and cosine :

$$x(t) = Acos(\omega_0 t) + Bsin\,(\omega_0 t)$$

Where $\omega_0 = \sqrt{k/m}$, and the constants A and B are chosen to match initial conditions.

For example if $x(0) = x_0$ and $v(0) = \left(\frac{dx}{dt}\right)\Big]_{t=0}$, then

$$A = x_0 \quad , B = 0$$

An equivalent way to write this solution is to put

$$x = Ae^{i(\omega_0 t + \phi_0)})$$

And match initial conditons by adjusting the constants A and ϕ_0 . With the initial conditions chosen above .we would put

.

$$\boxed{A = x_0 \text{ and } \phi_0 = 0}$$

This is easy to see:

$$\frac{dx}{dt} = i\omega_0 A e^{i(\omega_0 t + \phi_0)} = i\omega_0 x \qquad \dots (2)$$

$$\frac{}{dt^2} = -\omega^2 A e^{i(\omega_0 t + \phi_0)} = -\omega^2 x \qquad \dots (3)$$

Substitute eq. (3), (2) in to (1)

$$-\omega^2 m x + k x = 0$$

Then the solution satisfied

$$\omega_0 = \sqrt{k/m}$$

Complex Numbers

2022

Complex Functions

المحاضرة السابعة :

الدوال العقدية مع أمثلة تطبيقية

Complex Functions الدوال العقدية

Real part of f(z)

Imaginary part of f(z)

$$\boxed{w = f(z) = \boldsymbol{U(x,y) + iV(x,y)}}$$

The variable z is sometimes called an *independent* variable, w is called a *dependent* variable.

Example:

If $f(z) = z^2$, then find $f(z)$ if $z = 2i$

$$f(2i) = (2i)^2 = -4$$

Example

If $w = z^2$, then $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, and the transformation is

$$\boxed{\begin{aligned} f(z) &= z^2 = (x+iy)^2 \\ f(z) &= x^2 + 2xyi - y^2 \end{aligned}}$$

$$U = x^2 - y^2, \qquad V = 2xy$$

Example:

Illustrate the function

$$\underbrace{\sqrt{\qquad\qquad}}_{u(x,y)} \qquad \underbrace{\qquad)}_{v(x,y)}$$

$$\Longrightarrow \; u(x,y) = \sqrt{x^2 + y^2}$$

$$\Longrightarrow \; v(x,y) = -y$$

Elementary Functions

1. Polynomial function are defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z)$$

Where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and n is a positive integer called the degree of the polynomial $P(z)$.

Complex Exponential function

$$e \quad , i \quad , \pi$$

$$\textbf{Euler's formula}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

De'Moiver theorem

$$e^{i(n\theta)} = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

2. Rational Algebraic functions are defined by $w = \frac{P(z)}{Q(z)}$

Where $P(z), Q(z)$ are polynomials $Q(z) \neq 0$.

3. Exponential functions are defined by

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Where $e = 2.71828 \ldots$ is the natural base of logarithms .If a is real and positive, we define: $\boxed{a^z = e^{z \ln a}}$ (***prove***)

Where $\ln a$ *is the natural logarithm of* a.

Proof

$$w = a^z \Longrightarrow \ln w = \ln a^z \Longrightarrow \ln w = z \ln a$$

$$\therefore w = e^{z \ln a}$$

* دالة اسية مرفوعة الى (الاس مضروب في لوغاريتم الاساس)

Complex exponential functions have properties similar to those of real exponential function. For example,

$$e^{z_1} * e^{z_2} = e^{z_1+z_2},$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

4. Trigonometric functions (or circular functions) $\cos z, \sin z$, etc. can be defined in terms of exponential functions as follows.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{e^{iz} + e^{-iz}}{i(e^{iz} - e^{-iz})}$$

- $\sin^2 z + \cos^2 z = 1, \quad 1 + \tan^2 z = \sec^2 z,$
  $1 + \cot^2 z = \csc^2 z$
- $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$
  $\tan(-z) = -\tan z$
  $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
  $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

5. Hyperbolic Functions are defined as follows

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{\cosh z}{\sinh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

The following properties hold

- $\cosh^2 z - \sinh^2 z = 1, \quad 1 - \tan^2 z = \operatorname{sech}^2 z,$
  $\coth^2 z - 1 = \operatorname{csch}^2 z$
- $\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z,$
  $\tanh(-z) = -\tanh z$
- $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$
  $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$
- The following relations exist between the trigonometric or circular functions and the hyperbolic functions.

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z,$$

$$\tan iz = i \tanh z$$

$$\sinh iz = i \sin z, \cosh iz = \cos z,$$

$$\tanh iz = i \tan z$$

Logarithmic Functions:

If $z = e^w$, then we write $w = ln\, z$, called the natural logarithm of z

$$w = \ln z, \qquad z = re^{i\theta}$$

$$w = ln(re^{i\theta}) = ln\, r + ln\, e^{i\theta}$$

$$\boxed{w = \boldsymbol{lnz} = \mathbf{ln}\, \boldsymbol{r} + \boldsymbol{i}(\boldsymbol{\theta} + \mathbf{2}\boldsymbol{k\pi})}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Where $z = re^{i} \;\; = re^{i(\theta + 2k\pi)}$

$ln\, z \Longrightarrow$ is multiple-valued function. ***The principle branch*** of principle value of $ln\, z$ is defined as:

$$ln\, r + i\theta, \qquad \text{where } 0 \le \theta \le 2\pi$$

- The logarithmic function can be defined for real bases other than e. If

$$\boxed{z = a^w}, \qquad w = \log_a z$$

Where $a > 0$ and $a \neq 0,1$.

In this case $\boxed{z = e^{w \ln a}}$ and so $\boxed{w = (ln\, z)\, /\, ln\, a)}$

6. Inverse Trigonometric Functions (تعطى)

If $\boxed{z = \sin w}$, the $w = \sin^{-1} z$ is called the inverse sine of z: or arc sine of z.

$$w = \sin^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1 - z^2}\right)$, • $\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2 - 1}\right)$

7. Inverse Hyperbolic functions(تعطى)

If $= \sinh w$, the $w = \sinh^{-1} z$ is called the inverse hyperbolic sine of z.

$$w = \sinh^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$, $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

Example

1. Prove that:

a) $e^{z_1} * e^{z_2} = e^{z_1 + z_2}$

b) $|e^z| = e^x$

c) $e^{z + 2k\pi i} = e^z$, $k = 0, \pm 1, \pm 2, \ldots$

Solution

a) By definition

$$e^z = e^x(\cos y + i \sin y) \qquad \text{where} \begin{cases} z = x + iy \\ z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{cases}$$

$$\therefore e^z * e^z = e^{x_1}(\cos y_1 + i\sin y_1) * e^{x_2}(\cos y_2 + i\sin y_2) = e^{x_1} * e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) = e^{x_1+x_2}\cos(y_1 + y_2) + i\, e^{x_1+x_2} sin(y_1 + y_2)$$

b) $|e^z| = |e^x(\cos y + i\sin y)| = |e^x||\cos y + i\sin y|$

c) By part (a)

$$e^{z+2k\pi i} = e^z * e^{2k\pi i} = e^z(\cos 2k\pi + i\sin 2k\pi) = e^z$$

$2k\pi i \Longrightarrow$ is a ***period*** of the function ,In particular, $\boldsymbol{e^z}$ ***has*** ***<u>period</u>*** $\boldsymbol{2\pi i}$.

2. Find the value of $\boldsymbol{ln(1-i)}$. What is the principle value?

Or, determine the Principle value of $ln(1-i)$.

$$ln\, z = ln(re^{i\theta}) = ln\, r + i\theta$$

$z = 1 - i \Longrightarrow x = 1, y = -1,$

$$r = \sqrt{2}$$

$$\theta = -45°, \text{or} -\frac{\pi}{4}$$

$\ln z = \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right) =$

$\frac{1}{2}ln\, 2 + \frac{7\pi}{4}i + 2k\pi i$

The ***principle value*** is $(\boldsymbol{k = 0})$, $\left(\boldsymbol{\frac{1}{2}\ln 2 + \frac{7\pi}{4}i}\right)$

a) Find the principle value of i^i.

Solution

(a) by definition $z^i = e^{i\ln z} = e^{i\{\ln r + i(\theta+2k\pi)\}} = e^{i\ln r - (\theta+2k\pi)}$

$$= e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$$

$0 \le \theta \le 2\pi$

$$k = 0, \text{then } z^i = e^{-\theta}\{\cos(\ln r) + i\sin(\ln r)\}$$

(b) By definition, $i^i = e^{i\ln i}$, the principle value $= i^i = e^{-\pi/2}$

Now ,

Let $w = Z^Z$, $ln\, w = ZlnZ$

Example : $(1+i)^{(2-5i)}$

Method#1

Let , $w = (1+i)^{(2-5i)}$

$lnw = (2-5i)\,ln(1+i)$, $ln\,z = lnr + i\theta$,

$x = 1\,, y = 1\,, r = \sqrt{2}\,,\quad \theta = 45^0\,, or\ \frac{\pi}{4}$

$ln(1+i) = ln\sqrt{2} + i\frac{\pi}{4}$

$$lnw = (2-5i)\,ln(1+i) = (2-5i)[ln\sqrt{2} + i\frac{\pi}{4}]$$

$$lnw = 2ln\sqrt{2} + i\frac{\pi}{2} - 5ln\sqrt{2}\,i + \frac{5\pi}{4}$$

Then taking the exponential of both sides ……

Method#2

$$w = (1+i)^{(2-5i)}$$

$$w = e^{(2-5i)ln(1+i)} = e^{(2-5i)[ln\sqrt{2}+i\frac{\pi}{4}]} = e^{2ln\sqrt{2}+i\frac{\pi}{2}-5ln\sqrt{2}\,i+5\frac{\pi}{4}}$$

$$w = e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)+\boldsymbol{i}(\frac{\pi}{2}-5ln\sqrt{2})}$$

$$w = e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)} \times e^{\boldsymbol{i}(\frac{\pi}{2}-5ln\sqrt{2})}$$

Make use of Euler's formula

$$\boldsymbol{e^{i\theta} = cos\theta + isin\theta}$$

Solve : $i^{(2i)}\,, (1-\blacksquare)^{(2.5-1.3i)}\,,\ 10^{-7i}\,,\ (-9+\sqrt{3}\,i)^{(-6i)}$

(a) $w = i^{2i}$

$lnw = (2i)ln(i)$

$$\boxed{ln\,z = lnr + i\theta}$$

$$ln\,i = ln1 + i\frac{\pi}{2} = 0 + i\frac{\pi}{2}\ ,\quad ln\,i = i\frac{\pi}{2}$$

Then ,

$$lnw = (2i)\left[i\frac{\pi}{2}\right] = -\pi$$

$$ln\,w = -\pi$$

Taking ***exponential*** *for both sides:*

$$e^{lnw} = e^{-\pi} \rightarrow \quad w = e^{-\pi}$$

$$\boxed{w = e^{-\pi}}$$

(b) $(1-3i)^{(2.5-1.3i)}$

$$w = (1-3i)^{(2.5-1.3i)} \quad \rightarrow \; ln\,w = ln\left[(1-3i)^{(2.5-1.3i)}\right]$$

$$ln\,w = (2.5-1.35i)ln(1-3i)$$

$$\boxed{ln\,z = lnr + i\theta}$$

$ln(1-3i)$!!!!!!

$x = 1\,,\quad y = -3\;, r = 3.16,\;\; \theta = -7.1.6^o \;\; or\,, \theta = 288.4^o\;, \theta = 1.6\pi$

$$ln(1-3i) = ln(3.16) + 288.4^o\,\boldsymbol{i}$$

$$\therefore \;\; ln\,w = ln(3.16) + 288.4^o\,\boldsymbol{i} = 1.15 + 288.4^o\,i$$

Then ,

$$e^{lnw} = e^{1.15+288.4^o\,i}$$

$$w = e^{1.15} \times e^{288.4^o\,i}$$

$$(1-3i)^{(2.5-1.3i)} = 3.16[cos288.4^o + isin288.4^o]$$

$$(1-3i)^{(2.5-1.3i)} = (3.16 \times 0.316) - \boldsymbol{i}(3.16 \times 0.949)$$

$$\boxed{(1-3i)^{(2.5-1.3)} = 1.14 - 2.99\boldsymbol{i}}$$

Complex Numbers

2022

Complex Functions

أمثلة (أضافية) عن الدوال العقدية

Example 1 : Find the value of $ln(1-i)$. What is the principle value?

Or, determine the Principle value of $ln(1-i)$.

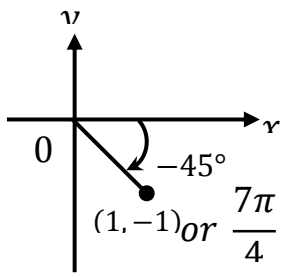
$$ln\, z = ln\left(re^{i\theta}\right) = ln\, r + i\theta \qquad \boldsymbol{\pi = 3.14}$$

$$z = 1 - i \implies x = 1, \qquad y = -1, \quad r = \sqrt{2}$$

$\theta = -45°$, or $\left(-\frac{\pi}{4}\right)$, or $(+\frac{7\pi}{4})$, or , $(-0.785$ or $5.49)$

$$ln\, z = \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right)$$

$$= \frac{1}{2} ln\, 2 + \frac{7\pi}{4} i + 2k\pi i$$



The ***principle value*** is $(\boldsymbol{k = 0})$, $\left(\boldsymbol{\frac{1}{2} ln\, 2 + \frac{7\pi}{4} i}\right)$

Or ,

$$ln(1-i) = 0.35 - 0.79\boldsymbol{i} \quad , \qquad or \quad , ln(1-i) = 0.35 + 5.49\boldsymbol{i}$$

Example 2: Find the value of e^{1+i}.

Solution

$$e^{1+i} = e^1 . e^i$$

باستخدام الحاسبة اليدوية ، فان

$$e^1 = 2.718$$

أما e^i ، والتي تمثل **دالة اسية عقدية** يتم استخدام صيغة أويلر (Euler's formula)

$$e^{i\theta} = cos\theta + isin\theta$$

$$e^i = \cos(1) + isin(1) = 0.54 + 0.84\boldsymbol{i}$$

Example 3:

سبق وان تم الاشارة في محاضرات سابقة بان هناك :

- ✓ **Power of $Z \Rightarrow$** Z^n , n is integer number may be positive or negative
- ✓ **Root of $Z \Rightarrow$** $Z^{1/n}$, n is integer and positive .

الان اذا كان (عدد عقدي او دالة عقدية) مرفوع (لعدد عقدي او دالة عقدية) أخر ى ... (سبق وان اعطينا الامثلة التالية) :

$\boldsymbol{Z^Z}$, $\boldsymbol{(1+i)^i, i^{2i}, 5^{-(1-2i)}}$, ...

Example 3 : $(1+i)^{(2-5i)}$

Method#1

Let , $w = (1+i)^{(2-5i)}$

$$lnw = (2-5i)\, ln(1+i) \qquad \ldots(1)$$

$$ln\, z = lnr + i\theta \ ,$$

$$x = 1\,, y = 1 \ ,$$

$$r = \sqrt{x^2+y^2} \ \Longrightarrow r \ \text{[redacted]} \ ,$$

$$\theta = tan^{-1}(y/x) \quad \Longrightarrow \theta = 45^0 \ , or \ \frac{\pi}{4} \ , or\ 0.785\ rad.$$

$$ln(1+i) = ln\sqrt{2} + i\frac{\pi}{4}$$

Or ,

$$ln(1+i) = 0.346 + 0.785\boldsymbol{i}$$

وبتعويض هذا الناتج في المعادلة (1) نجد ان

$$lnw = (2-5i)\, ln(1+i) = (2-5i)[ln\sqrt{2} + i\frac{\pi}{4}]$$

$$lnw = 2ln\sqrt{2} + i\frac{\pi}{2} - 5ln\sqrt{2}\, i + \frac{5\pi}{4}$$

Then taking the exponential of both sides ……

Method#2

$$w = (1+i)^{(2-5i)}$$

$$w = e^{(2-5i)\,ln(1+i)} \qquad \dots (1)$$

$$w = e^{(2-5i)[ln\sqrt{2}+\boldsymbol{i}\frac{\pi}{4}]} = e^{(2-5i)[0.346+0.785\boldsymbol{i}]}$$

$$w = e^{0.692} \times e^{1.57\boldsymbol{i}} \times e^{-1.73\boldsymbol{i}} \times e^{3.925}$$

$$w = e^{4.617} \times e^{-0.16\boldsymbol{i}}$$

باستخدام الحاسبة اليدوية لانها دالة حقيقية

تحسب باستخدام صيغة اويلر(Euler's formula) لانها دالة اسية عقدية ،

$$\boldsymbol{e^{i\theta} = cos\theta + isin\theta}$$

$$w = 101.19[cos0.16 - \boldsymbol{i}sin0.16]$$

راجع الملاحظة السابقة في طريقة ايجادها ،

$$w = 99.9 - 16.13\boldsymbol{i}$$

Example 4: Evaluate $sin(i)$

Solution

Method#1:

$$\boxed{sin\,z = \frac{e^{iz} - e^{-iz}}{2i}}$$

بالمقارنة مع المسالة نجد ان $z = i$

$$\therefore \quad sin\,z = \frac{e^{iz}-e^{-iz}}{2i} = \frac{i}{2i^2}[e^{iz} - e^{-iz}]$$

بضرب البسط والمقام ب (i)

Then ,

$$sin\,z == \frac{-i}{2}[e^{iz} - e^{-iz}]$$

$$\therefore \quad sin(i) = \left(\frac{-i}{2}\right)\left[e^{i(i)} - e^{-i(i)}\right] = \left(\frac{-i}{2}\right)\left[e^{-1} - e^{+1}\right]$$

(لاحظ ان e^{+1} ، e^{-1} دوال اسية حقيقية يمكن ايجاد قيمتها باستخدام الحاسبة اليدوية) .

$$e^{+1} = 2.718\,,$$

$$e^{-1} = \frac{1}{e^{1}} = \frac{1}{2.718} = \mathbf{0.367}$$

$$\therefore \quad sin(i) = \left(\frac{-i}{2}\right)\left[e^{i(i)} - e^{-i(i)}\right] = \left(\frac{-i}{2}\right)[0.367 - 2.718] = 1.18\boldsymbol{i}$$

$$\therefore \quad sin(i) = 1.18\boldsymbol{i}$$

Example 5: Evaluate $sinh(1+i)$

Solution

$$\boxed{sinhz = \frac{e^{Z} - e^{-Z}}{2}}$$

$$sinh(1+i) = \frac{e^{(1+i)} - e^{-(1+i)}}{2} = \frac{1}{2}\left[e^{1} \times e^{i} - e^{-1} \times e^{-i}\right]$$

$$sin\,h(1+i) = \frac{1}{2}[2.718(0.54 + 0.84i) - 0.367(0.54 - 0.89i)] = 0.635 + 1.30i$$

$$sin\,h(1+i) = 0.635 + 1.29i$$

Now,

$$\boldsymbol{i}sin\,h(1+i) = -1.29 + 0.635\boldsymbol{i}$$

Activity :

1. Evaluate $cos(i)$
2. Evaluate $cos(1+i)$
3. Evaluate $cos\,h(i)$
4. Evaluate $cosh(1+i)$

Complex Numbers

2022

Complex Derivative

المحاضرة الثامنـــة :

مشتقة الدوال العقدية وتطبيقاتها

Derivative of complex function were defined in terms of *limit* and the other in terms of *formulas* ;

أي لغرض ايجاد مشتقة الدالة العقدية فان هناك طريقتين :

- ✓ الاولى هي استخدام (**الغايات**) و
- ✓ الثانية استخدام (**قواعدالمشتقات**)

وهو نفس الاسلوب المتبع سابقا في ايجاد مشتقة الدوال في حسابان التكامل والتفاضل للدوال الحقيقية وكالاتي :

$$f'(Z_0) = \frac{\Delta w}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{f(Z_0 + \Delta Z) - f(Z_0)}{\Delta Z}$$

Or,

$\frac{dw}{dZ} = f'(Z)$	**1st** derivative of w with respect to z
$\frac{d^2w}{dZ^2} = \frac{d}{dz}(\frac{dw}{z}) = f''(Z)$	**2nd** derivative of w with respect to z
$\frac{d^nw}{dZ^n} == f^{(n)}(Z)$	**nth** derivative of w with respect to z

Rules for differentation

If $f(z), g(z)$ and $h(z)$ are " *analytic* " function of z the following differentation rules are *valid* :

Complex Derivative

2022

1.	$\boldsymbol{\frac{d}{dz}[f(z)+g(z)] = \frac{d}{dz}f(z) + \frac{d}{dz}g(z)}$ $\boldsymbol{= f'(z) + g'(z)}$
2.	$\frac{d}{dz}[f(z)-g(z)] = \frac{d}{dz}f(z) - \frac{d}{dz}g(z)$ $= f'(z) - g'(z)$
	$\frac{d}{dz}\{cf(z)\} = c\frac{d}{dz}f(z)$, and C is constant
4.	$\frac{d}{dz}[f(z)g(z)] = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$ $= f(z)g'(z) + g(z)f'(z)$
	$\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$ $= \frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$, if $g(z) \neq 0$

والجدول التالي يوجز مشتقات اهم الدوال :

2022

$\frac{d}{dz}z^a = az^{a-1}$	$\frac{d}{dz}z^z = z^z(1+\ln z)$
$\frac{d}{dz}a^z = a^z\ln(a)$	$\frac{d}{dz}\ln z = \frac{1}{z}$
$\frac{d}{dz}e^z = e^z$	$\frac{d}{dz}\log_a z = \frac{1}{z\ln a}$

$\frac{d}{dz}\sin(z) = \cos(z)$	$\frac{d}{dz}\sinh(z) = \cosh(z)$
$\frac{d}{dz}\cos(z) = -\sin(z)$	$\frac{d}{dz}\cosh(z) = \sinh(z)$
$\frac{d}{dz}\tan(z) = \sec^2(z)$	$\frac{d}{dz}\tanh(z) = 1-\tanh^2(z) = \operatorname{sech}^2(z)$
$\frac{d}{dz}\cot(z) = -\csc^2(z)$	$\frac{d}{dz}\coth(z) = -\operatorname{csch}^2(z)$
$\frac{d}{dz}\csc(z) = -\csc(z)\cot(z)$	$\frac{d}{dz}\operatorname{csch}(z) = -\operatorname{csch}(z)\coth(z)$
$\frac{d}{dz}\sec(z) = \sec(z)\tan(z)$	$\frac{d}{dz}\operatorname{sech}(z) = -\operatorname{sech}(z)\tanh(z)$

$\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$	$\frac{d}{dz}\sinh^{-1}(z) = \frac{1}{\sqrt{1+z^2}}$
$\frac{d}{dz}\cos^{-1}(z) = -\frac{1}{\sqrt{1-z^2}}$	$\frac{d}{dz}\cosh^{-1}(z) = \frac{1}{\sqrt{z+1}\sqrt{z-1}}$
$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$	$\frac{d}{dz}\tanh^{-1}(z) = \frac{1}{1-z^2}$
$\frac{d}{dz}\cot^{-1}(z) = -\frac{1}{1+z^2}$	$\frac{d}{dz}\coth^{-1}(z) = \frac{1}{1-z^2}$
$\frac{d}{dz}\csc^{-1}(z) = -\frac{1}{z\sqrt{z^2-1}}$	$\frac{d}{dz}\operatorname{csch}^{-1}(z) = -\frac{1}{z^2\sqrt{1+\frac{1}{z^2}}}$
$\frac{d}{dz}\sec^{-1}(z) = \frac{1}{z\sqrt{z^2-1}}$	$\frac{d}{dz}\operatorname{sech}^{-1}(z) = -\frac{1}{z(z+1)\sqrt{\frac{1-z}{1+z}}}$

Example 1 : Find the 1st derivative for the function

$$f(z) = [z + (z^2+1)^2]^2$$

$$\frac{d}{dz}[z+(z^2+1)^2]^2 = 2[z+(z^2+1)^2]\frac{d}{dz}[z+(z^2+1)^2]$$
$$= 2[z+(z^2+1)^2][1+2(z^2+1)\times 2z]$$

At $z = 1+i$, then

2022 .....

$$\frac{d}{dz}[z+(z^2+1)^2]^2$$

$$= 2[(1+i)+\{(1+i)^2+1\}^2]\times[1+2\{(1+i)^2+1)\times 2(i)\}]$$

$$= \ldots\ldots$$

Example 2 : Find the 1st derivative for the function

$$f(z)=\left[\frac{(z+2i)(i-z)}{(2z-1)}\right] \quad ,at \quad z=i$$

Solution

$$(z+2i)(i-z)=zi-z^2-2-2iz=-2-iz-z^2$$

$$\frac{d}{dz}\left[\frac{(z+2i)(i-z)}{(2z-1)}\right]=\frac{d}{dz}\left[\frac{-2-iz-z^2}{(2z-1)}\right]=\frac{(2z-1)(-i-2z)-(-2-iz-z^2)(2)}{(2z-1)^2}$$

Example 3 : Find the 1st derivative for the function $[z+(z^2+1)^2]^2$

$$\frac{d}{dz}[z+(z^2+1)^2]^2=2\{z+(z^2+1)^2\}*[1+2(z^2+1)(2z)]$$

Example 4 : Find the 1st derivative for the function

$$f(z)=(z+2\sqrt{z})^{1/3}$$

$$\frac{d}{dz}\left(z+2\sqrt{z}\right)^{1/3}=\frac{1}{3}[(z+2\sqrt{z})^{-2/3}]*\left(1+2\left(\frac{1}{2}\right)z^{-1/2}\right]=\ldots\ldots.$$

Example 5 : Find $\frac{d}{dz}(1+z^2)^{3/2}$

$$\frac{d}{dz}(1+z^2)^{\frac{3}{2}}=\frac{3}{2}(1+z^2)^{1/2}\times\frac{d}{dz}(1+z^2)$$

$$\frac{d}{dz}(1+z^2)^{3/2}=\frac{3}{2}(1+z^2)^{1/2}\times(2z)=\cdots\ldots$$

Example 6 : Find $\frac{d}{dz}\left[\frac{(z+2i)(i-z^2}{(2z-1)^2}\right] at \quad z=i$

2022

Example 7 : Find 1st derivative for the function z^{lnz}

Let , $w = z^{lnz}$

$$lnw = (lnz) \times l\,n(z) \rightarrow \quad lnw = (ln\,z)^2$$

$$\frac{1}{w}\frac{dw}{dz} = 2ln\,z\left(\frac{1}{z}\right) \rightarrow \frac{dw}{dz} = w \times 2ln\,z \times \left(\frac{1}{z}\right)$$

$$\therefore \frac{dw}{dz} = z^{lnz} \times 2lnz \times z^{-1}$$

$$\frac{dw}{dz} = 2(z^{lnz-1})lnz$$

Example 8 : Find $\frac{dw}{dz}$ for the function z^i

Let , $w = z^i$

$$lnw = ln(z^i) \rightarrow \quad lnw = i(lnz)$$

$$\frac{1}{w}\frac{dw}{dz} = i\left(\frac{1}{z}\right) \rightarrow \frac{dw}{dz} = i\left(\frac{1}{z}\right) * w$$

$$\therefore \frac{dw}{dz} = w \times \frac{i}{z} = \left(\frac{i}{z}\right)(z^i) = iz^{i-1}$$

Example 9 : Find $\frac{dw}{dz}\{sin(iz-2)^{tan^{-1}(z+3i)}\}$

Let , $w = sin(iz - \blacksquare)^{tan^{-1}(z+3i)}$

$$lnw = tan^{-1}(z+3i) \times ln\{sin(iz-2)\}$$

$$\frac{1}{w}\frac{dw}{dz} = tan^{-1}(z+3i) \times \frac{1}{sin(iz-2)} \times cos(iz-2) \times i + ln\{sin(iz-2) \times \frac{1}{1+(z+3i)^2} \times (1)\}$$

$$\frac{1}{w}\frac{dw}{dz} = w \times \dots\dots$$

Example 10 : Find $\frac{d}{dz} ln[z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\,]$

$$\frac{d}{dz} ln\left[z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}\right] = \frac{1}{z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}} \frac{d}{dz}[z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}]$$

$$= \frac{1}{z - \frac{3}{2} + \sqrt{z^2 - 3z + 2i}} [1 + \frac{1}{2}(z^2 - 3z + 2i)^{-1/2}(2z - 3)]$$

Example 11 : Find $\frac{d}{dz} tan^{-1}(z + 3i)^{-1/2}$

$$\frac{d}{dz} tan^{-1}(z + 3i)^{-1/2} = \frac{1}{1 + [(z + 3i)^{-1/2}]^2} \times \frac{d}{dz}(z + 3i)^{-1/2}$$

$$= \frac{1}{1+\left(\frac{1}{\sqrt{z+3i}}\right)^2}(-\frac{1}{2})(z + 3i)^{-\frac{3}{2}} \times (1)$$

Example 12 : Find $\frac{d}{dz} cos^{-1}(sinz - cosz)$

$$\frac{d}{dz} cos^{-1}(sinz - cosz) = \frac{-1}{\sqrt{1-(sinz-cosz)^2}} \frac{d}{dz}(sinz - cosz)$$

Example 13 : Find $\frac{d}{dz} sinh(z + 1)^2$

$$\frac{d}{dz} sinh(z + 1)^2 = cosh(z + 1)^2 \times 2(z + 1)(1)$$

Example 14 : Find $\frac{d}{dz} cosh^{-1}(lnz)$

$$\frac{d}{dz} cosh^{-1}(lnz) = -\frac{1}{\sqrt{1 - (lnz)^2}} \times \frac{1}{z}$$

Activity : Find $\frac{d}{dz} 5^{(z^2+i)}$, *Hint* $\frac{d}{dz} a^z = a^z \times lna \times \frac{d}{dz}(z)$

ثم **قارن** مع مثال7 ، 8 و 9 **ماذا تستنتج !!!**

$$\frac{d}{dz} 5^{(z^2+i)} = 5^{(z^2+i)} \times ln5 \times \frac{d}{dz}(z^2 + i) = 5^{(z^2+i)} \times ln5 \times (2z)$$

Complex Function Separation

المحاضرة 9

تجزئــة الدوال العقدية الى اجزائها الحقيقية والخيالية

Complex function $w = f(z) = U(x,y) + \boldsymbol{i}\, V(x,y)$

Example 1: Separate each of the following functions into real and imaginary parts , *or in other words*

Find :

$U(x,y)$ *and* $V(x,y)$ such that $f(z) = U + \boldsymbol{i}\, V$

(a) $f(z) = z$

(b) $f(z) = z^2 - i$

(c) $f(z) = 2z^2 - 3iz$

(d) $f(z) = z + \frac{1}{z}$

(e) $f(z) = \frac{z^2+1}{z}$

(f) $f(z) = \frac{1-z}{1+z}$

(g) $f(z) = \sqrt{z}$

(h) $f(z) = lnz$

(i) $f(z) = e^{3iz}$

(j) $f(z) = sin2z$

(k) $f(z) = cosz$

(l) $f(z) = z^z$

(m) $f(z) = z^2 e^{2z}$

(n) $f(z) = sinhz$

Solution

(a) $f(z) = z$

$$f(z) = x + iy \;\rightarrow\; u(x,y) = x \;, v(x,y) = y$$

(b) $f(z) = z^2 - i$

$$f(z) = (x+iy)^2 - i$$

$$f(z) = x^2 + 2xyi - y^2 - i$$

2022

$$f(z) = (x^2 - y^2) + i(2xy - 1)$$

$\boldsymbol{U(x,y)}$ $\boldsymbol{V(x,y)}$

(c) $f(z) = 2z^2 - 3iz$

$$f(z) = 2(x+iy)^2 - 3i(x+iy)$$

$$f(z) = 2[x^2 + 2xyi - y^2] - 3xi + 3y$$

$$f(z) = 2x^2 + 4xyi - 2y^2 - 3xi + 3y$$

$$f(z) = \underbrace{(2x^2 - 2y^2 - 3y)}_{\boldsymbol{U(x,y)}} + \boldsymbol{i}\,\underbrace{(4xy - 3x)}_{\boldsymbol{V(x,y)}}$$

(d) $f(z) = z + \frac{1}{z}$ *the same as* $(\boldsymbol{e})$ $f(z) = \frac{z^2+1}{z}$

Method#1

$$f(z) = (x+iy) + \frac{1}{(x+iy)} = (x+iy) + \frac{1}{(x+iy)} \times \frac{x-iy}{x-iy}$$

$$f(z) = z + \frac{1}{z} = x + iy + \frac{x-iy}{x^2+y^2}$$

$$f(z) = x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$$

$$f(x+iy) = \underbrace{\left[\frac{x(x^2+y^2)+x}{x^2+y^2}\right]}_{\boldsymbol{U(x,y)}} + i\underbrace{\left[\frac{y(x^2+y^2)-y}{x^2+y^2}\right]}_{\boldsymbol{V(x,y)}}$$

Method#2

$$f(z) = (x+iy) + \frac{1}{(x+iy)} = \frac{(x+iy)^2+1}{x+iy}$$

$$z + \frac{1}{z} = \frac{(x^2-y^2+1)+2xyi}{x+iy} \times \frac{x-iy}{x-iy}$$

$$f(z) =$$

Method#3

This problem can be sovled with the aid of complex conjugate of z , i.e.

$f(z) = z + \frac{1}{z}$

$$f(z) = \frac{z^2+1}{z} \quad \rightarrow \quad f(z) = \frac{z^2+1}{z} \times \frac{\bar{z}}{\bar{z}}$$

$$f(z) = \frac{z^2\bar{z} + \bar{z}}{z\bar{z}} = \frac{z^2\bar{z} + \bar{z}}{|z|^2}$$

Then substitute $z = x + iy$ in above equation ...what about the result. Disucss!!.

(f) $f(z) = \frac{1-z}{1+z}$

$$f(x+iy) = \frac{1-(x+iy)}{1+(x+iy)} = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy)}{(1+x)-iy)}$$

$$= \frac{(1-x)(1+x) - iy(1+x) - iy(1-x) - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - iy - xyi - iy + xyi - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - 2iy - y^2}{(1+x)^2 + y^2}$$

$$f(z) = \frac{1-x^2-y^2}{(1+x)^2+y^2} - \boldsymbol{i}\frac{2y}{(1+x)^2+y^2}$$

$\boldsymbol{U(x,y)}$ $\boldsymbol{V(x,y)}$

(g) $f(z) = \sqrt{z}$

$$f(x+iy) = (x+iy)^{1/2}$$

Or in polar form , since $z = re^{i\theta} \rightarrow \quad z = r[cos\theta + isin\theta]$

Then ,

$$z^{1/n} = r^{1/n}[\cos\left(\frac{\theta + 2k\pi}{n}\right) + isin\left(\frac{\theta + 2k\pi}{n}\right)]$$

$k = 0$, $n = 2$, $\quad r = \sqrt{x^2 + y^2}$, $\quad \theta = tan^{-1}(y/x)$

$$\therefore \ \sqrt{z} = z^{1/2} = (x^2 + y^2)^{1/2}[\cos\left(\frac{\theta}{2}\right) + isin\left(\frac{\theta}{2}\right)]$$

Where

$$x = rcos\theta \ , \ cos\theta = \frac{x}{r}$$

$$y = rsin\theta \ , \ sin\theta = \frac{y}{r}$$

$$\therefore \ \sqrt{z} = z^{1/2} = r^{1/2}\left[\frac{x}{2r} + i\,\frac{y}{2r}\right] = \frac{1}{2\sqrt{r}}[x + iy]$$

$$\therefore \ z^{1/2} = \frac{1}{2(x^2 + y^2)^{1/2}}[x + iy] = \frac{x}{2(x^2 + y^2)^{1/2}} + i\,\frac{y}{2(x^2 + y^2)^{1/2}}$$

$$\boldsymbol{U(x,y)} \qquad \boldsymbol{V(x,y)}$$

(h) $\quad f(z) = lnz$

Since , $\quad lnz = lnr + i\theta$

$$U(x,y) = lnr$$

$$V(x,y) = \theta$$

Since

$$r = \sqrt{x^2 + y^2} \ , \quad \theta = tan^{-1}(y/x)$$

$$lnz = ln(x^2 + y^2)^{1/2} + \boldsymbol{i}\, tan^{-1}(y/x)$$

$$lnz = \frac{1}{2}ln((x^2 + y^2) + \boldsymbol{i}\, tan^{-1}(y/x)$$

$\therefore \quad U(x,y) = \frac{1}{2}ln((x^2 + y^2)$ and $\quad V(x,y) = tan^{-1}(y/x)$

(i) $\boldsymbol{f(z) = e^{3iz}}$

$$e^{3iz} = e^{3i(x+iy)} = e^{3ix-3y} = e^{3ix} \times e^{-3y}$$

$$= e^{-3y}[\cos(3x) + \boldsymbol{i}\, sin(3x)]$$

$\therefore \quad e^{3iz} = e^{-3y}\cos(3x) + \boldsymbol{i}\, e^{-3y} sin(3x)]$

$\therefore \quad U(x,y) = e^{-3y}\, cos(3x) \quad \& \quad V(x,y) = e^{-3y} sin(3x)$

(j) $f(z) = sin(2z)$

$f(x+iy) = sin\,2(x+iy) = \sin(2x+2iy)$

$= sin(2x)\cosh(2y) + \boldsymbol{i}\, cos(2x)\ sin(2y)$

$sin(2z) = sin(2x)\, cosh(2y) + cos(2x)\, sinh(2y)$

Where ,

$$\boxed{sin(iz) = \boldsymbol{i} sinh(z)}$$

(k) $f(z) = sin(2z)$

Method#1

Let , $f(z) = cosz$

$f(x+iy) = \cos(x+iy)$

باستخدام جيب تمام حاصل جمع زاويتين ، فان

$$\mathrm{Cos}(x+iy) = cosxcos(iy) - sinxsin(iy)$$

ذكرنا سابقا في موضوع الدوال العقدية ، ان هناك علاقة تربط بين الدوال المثلثية (جيب وجيب تمام) بالدوال الزائدية ، اي

$$\boxed{\begin{aligned} sin(iy) &= isinhy \\ cos(iy) &= coshy \end{aligned}}$$

Method#2

$f(z) = cosz$

كما هو معلوم،

$$cosz = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

وبالتعويض عن $z = x + iy$ بالمعادلة اعلاه ، نحصل على ،

$$cosz = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\left[e^{ix}.e^{-y} + e^{-ix}.e^{y}\right]$$

$$= \frac{1}{2}[e^{-y}(cosx + isinx) + e^{y}(cosx - isinx)]$$

$$= \frac{1}{2}[\boldsymbol{e^{-y}cosx} + ie^{-y}sinx + \boldsymbol{e^{y}cosx} - ie^{y}sinx]$$

الان نجمع الاجزاء الحقيقية مع بعضها والخيالية مع بعضها ، فتصبح ،

$$cosz = \frac{1}{2}[(\, e^{-y}\boldsymbol{cosx} + e^{y}\boldsymbol{cosx}) + i(e^{-y}\boldsymbol{sinx} - e^{y}\boldsymbol{sinx})]$$

ناخذ عامل مشترك بين الاجزاء الحقيقية و الاجزاء الخيالية ،نحصل على

$$cosz = \frac{1}{2}[cosx(e^{y} + e^{-y}) - isinx\,(e^{y} - e^{-y})]$$

بما ان ،

$$sinhz = \frac{1}{2}(e^{y} - e^{-y})$$

$$coshz = \frac{1}{2}(e^{y} + e^{-y})$$

وعليه فان ، $cosz = cosxcoshy - \boldsymbol{i}\, sinxsinhy$

(l) $f(z) = z^{z}$

Let , $w = z^{z}$

$$lnw = l\,n(z^{z}) = zln(z)$$

$$w = e^{zlnz}$$

Then ,

$$w = e^{(x+iy)\ln(x+iy)}$$

Since ,

$$ln(z) = lnr + i\theta$$

Or,

$$ln(z) = ln\sqrt{x^2 + y^2} + \boldsymbol{i}tan^{-1}(y/x)$$

$$w = e^{(x+iy)[\frac{1}{2}ln(x^2+y^2)+\boldsymbol{i}\, tan^{-1}(y/x)}$$

$$\therefore \quad w = e^{\frac{1}{2}x\,ln(x^2+y^2)-y\,tan^{-1}(y/x)} \times e^{\boldsymbol{i}\,[\frac{y}{2}ln(x^2+y^2)+x\,tan^{-1}(y/x)}$$

(m) $f(z) = z^2e^{2z}$

$$f(x+iy) = (x+iy)^2e^{2(x+iy)}$$

$$= (x^2 - y^2 + 2ixy)e^{2x} * \; e^{2yi}$$

$$= e^{2x}[(x^2 - y^2 + 2ixy)] \times [\cos(2y) + isin(2y)]$$

$$z^2e^{2z} = e^{2x}(x^2 - y^2)\cos(2y) + \boldsymbol{i}\; e^{2x}(x^2 - y^2)sin(2y)]$$

$$+ \boldsymbol{i}\; e^{2x}(2xy)\cos(2y) - e^{2x}(2xy)\sin(2y)$$

Analytic (Cauchy –Remann Conditions)
and Harmonic complex functions (Laplace Equation)

المحاضرة 10

✓ **الدوال العقدية التحليلية و (شروط كوشي- ريمان)**

✓ **الدوال العقدية التوافقية (معادلة لابلاس)**

Previous lecture ,

Complex function $\boldsymbol{w = f(z) = U(x,y) + i\,V(x,y)}$

- Complex function to be *analytic* must satify *Cauchy – Remann Conditions* , that is :

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Example 1: Does the function $f(z) = e^z$ analytic ?.

Solution

$$f(z) = e^z = e^{x+iy} = e^x \times e^{i\,y}$$

$$e^z = e^x[cosy + \boldsymbol{i}siny] = e^x cosy + \boldsymbol{i}\, e^x\, siny$$

$$\therefore \quad u(x,y) = e^x cosy \quad , \quad v(x,y) = e^x\, siny$$

C-R-C's

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$

$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \, e^x cosy$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *analtyic*

- A complex function to be *harmonic* must satify *Laplace equation* , that is :

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad , or \quad U_{xx} + U_{yy} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad , or \quad V_{xx} + V_{yy} = 0$$

Or ,

$$\nabla^2 \psi = 0 \quad where \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The ' *operator* ' ∇^2 is called " *Laplacian*" .

Example 2 : Does the function $f(z) = e^z$ harmonic ?

Solution

$$\therefore \quad u(x,y) = e^x cosy \quad , v(x,y) = e^x \, siny$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$

$$\frac{\partial^2 U}{\partial x^2} = e^x cosy \quad , \quad \frac{\partial^2 V}{\partial x^2} = e^x \, siny$$

$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \, e^x cosy$$

$$\frac{\partial^2 U}{\partial y^2} = -e^x cosy \quad , \quad \frac{\partial^2 V}{\partial y^2} = -e^x \, siny$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *harmonic function .*

Example 3 : Does the function $f(z) = cosz$ analytic , harmonic ?

Solution

$$f(z) = cosz = cosx\ coshy - \boldsymbol{i}\ sinx\ coshy$$

$$\therefore \quad u(x,y) = cosx\ coshy \quad , v(x,y) = -sinx\ coshy$$

$$\frac{\partial U}{\partial x} = -sinx\ coshy \quad , \quad \frac{\partial V}{\partial x} = -cosx\ sinhy$$

$$\frac{\partial^2 U}{\partial x^2} = -cosx\ coshy \quad , \quad \frac{\partial^2 V}{\partial x^2} = sinx\ sinhy$$

$$\frac{\partial U}{\partial y} = cosx\ sinhy \quad , \quad \frac{\partial V}{\partial y} = -sinx\ coshy$$

$$\frac{\partial^2 U}{\partial y^2} = cosx\ coshy \quad , \quad \frac{\partial^2 V}{\partial y^2} = -sinx\ sinhy$$

The conditons are ***satisfied*** the the function $f(z) = cosz$ is *analytic* and *harmonic function .*

Example 4 : Does the function $f(z) = lnz$ analytic , harmonic ?

Note : $\frac{d}{dz} tan^{-1} z = \frac{-1}{1+z^2}$

Solution

$$f(z) = lnz = ln(x + \boldsymbol{i}y)$$

$$lnz = \frac{1}{2} ln(x^2 + y^2) + \boldsymbol{i}\ tan^{-1}(y/x)$$

$$\therefore U(x,y) = \frac{1}{2} l\,n(x^2 + y^2)$$

$$\frac{\partial U}{\partial x} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2y = \frac{y}{x^2 + y^2}$$

$$V(x,y) = tan^{-1}(y/x)$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+(y/x)^2} \times (-x^2 y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial V}{\partial y} = \frac{1}{1+(y/x)^2} \times (-x^{-1})$$

Activity :

Does the functions

1. $f(z) = z + \bar{z}$ analytic , harmonic ?
2. $f(z) = \frac{1}{z}$ analytic , harmonic ?
3. $f(z) = \frac{z+i}{z-i}$ analytic , harmonic ?
4. $f(z) = zlnz$ analytic , harmonic ?
5. $f(z) = \boldsymbol{i}\, cosz$ analytic , harmonic?

Complex Power Series Expansion

المحاضرة 11

متسلسلات الدوال العقديــة

- ✓ متسلسلة تايلــر (Taylor Series) ومتسلسلة ماكلورين (Maclaurin Series)
- ✓ متسلسلة لورنت (Laurent Series)

Taylor Series

Suppose that a function f is analytic throughout a circle $|z - z_0| < R$, centered at z_0 and with radius R_0 . Then $f(z)$ has the power series representation.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots . + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

Where ,

$$n! = n(n-1)(n-2) \ldots .3.2.1$$

$$0! = 1 \ , \ 1! = 1 \ , \ 2! = 2 * 1 = 2 \ ,$$

$$3! = 3 * 2 * 1 = 6 \, , \ldots . etc$$

ملاحظــة : عنما تكون $z = 0$ فان متسلسلة **تايلر** تؤول الى متسلسلة **ماكلورين** .

Here some function expansion and convergent region:

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < \infty$
2. $sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ \ldots \qquad |z| < \infty$
3. $cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ \ldots \qquad |z| < \infty$
4. $coshz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ \ldots \qquad |z| < \infty$

5. $(1+z)^{\propto} = 1+\propto z - \frac{\propto(\propto-1)z^2}{2!} + \cdots \qquad |z| < 1$

is the ***binomial theorem*** *for* $\propto = -1$ ***gives formula 8.***

6. lnz ***undefined*** *at* $z = 1$

7. $ln(1+z) = z - \frac{z2}{2} + \frac{z^3}{3} - \ldots\ldots \qquad |z| < 1$

8. $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad |z| < 1$

9. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$

The first five expansion are valid for all z , whilst the last three are only valid for $|z| < 1$.

Example 1 Expand $f(z) = ln(1+z)$

Solution

$$f(z) = ln(1+z) \quad , \quad f(0) = 0$$

$$f'(z) = \frac{1}{1+z} \quad , \quad f'(0) = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \quad , \quad f''(0) = -1$$

$$f^{(3)}(z) = \frac{(-1)(-2)}{(1+z)^2} \quad , \quad f^{(3)}(0) = 2$$

.....

.....

$$f^{(n+1)}(z) = \frac{(-1)^n \; n!}{(1+z)^{(n+1)}} \quad , \quad f^{(n+1)}(0) = (-1)^n \; n!$$

$$f(z) = ln(1+z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$f(z) = ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

Example 2 Expand $f(z) = ln\frac{(1+z)}{(1-z)}$

Solution

$$f(z) = ln\frac{(1+z)}{(1-z)} \quad ,$$

$$f(z) = ln\frac{(1+z)}{(1-z)} = ln(1+z) - ln(1-z)$$

$$ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1+z) - ln(1-z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots. + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots. l$$

$$= 2z + \frac{2z^3}{3} + \cdots. = 2(z + \frac{2z^3}{3} + \frac{z^5}{5} \ldots.) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

$$ln\frac{(1+z)}{(1+z)} = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

Example 3 Expand $f(z) = \frac{1}{(1+z)^2}$

Solution

$$f(z) = \frac{1}{(1+z)^2}$$

$$f(z) = \frac{1}{(1+z)^2} = (1+z)^{-2} \ , \quad f(0) = 1$$

$$f'(z) = \frac{-2}{(1+z)^3} \quad , \quad f'(0) = -2$$

$$f''(z) = \frac{6}{(1+z)^4} \quad , \quad f''(0) = 6$$

$$\therefore \quad \frac{1}{(1+z)^2} = 1 - 2z + 3z^2 + \cdots \qquad provided \quad |z| < 1$$

Example 4 Expand $f(z) = \frac{1}{1+z}$

Solution

$$f(z) = \frac{1}{1+z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1+z} \quad , f(0) = 1$$

$$f'(z) = \frac{-1}{(1+z)^2} \quad , \quad f'(0) = -1$$

$$f''(z) = \frac{2}{(1+z)^3} \quad , \quad f''(0) = 2$$

.....

.....

$$\therefore \quad \frac{1}{1+z} = 1 - z + z^2 + \cdots$$

Example 5 If $|z| < 1$, expand $f(z) = \frac{1}{1-z}$

Solution

$$f(z) = \frac{-1}{1-z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1-z} \quad , f(0) = 1$$

$$f'(z) = \frac{1}{(1-z)^2} \quad , \quad f'(0) = 1$$

$$f''(z) = \frac{2}{(1-z)^3} \quad , \quad f''(0) = 2$$

.....

.....

$$\therefore \quad \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \quad |z| < 1$$

Example 6 Expand $f(z) = e^{i\theta}$

Solution

$$\because \quad \boldsymbol{e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

Then

$$e^{i\theta} = 1 + (i\theta) + + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + \boldsymbol{i\theta} - \frac{\theta^2}{2!} - \boldsymbol{i}\frac{\boldsymbol{\theta^3}}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \boldsymbol{i}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\theta^3}}{3!} + \cdots\right)$$

$$e^{i\theta} = cos\theta + isin\theta \qquad \text{"Euler formula"}$$

Activity : $e^{i\pi} + 1 = 0$

Is called the most beautiful equation in all of mathematics

- ✓ It is an *identify* that contains the most beautiful entities encountered in math, namely $\boldsymbol{\pi}, \boldsymbol{i}, \boldsymbol{e}, \mathbf{0}$ and $\mathbf{1}$.
- ✓ It **combines** the real and the imaginary.

Complex Power Series Expansion

2022

In mathematics,***Euler's identity*** (also known as ***Euler's equation***) is the equality

$$e^{i\pi} + 1 = 0$$

where

e	is Euler's number, the base of natural logarithms,
i	is the imaginary unit, which by definition satisfies $i^2 = -1$, and
π	is pi, the ratio of the circumference of a circle to its diameter.

Euler's ***identity*** is a special case of *Euler's* ***formula***, which states that for any real number x,

$$e^{ix} = \cos x + i\sin x$$

where the inputs of the trigonometric functions sine and cosine are given in ***radians.*** In particular, when $x = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

Since

$$\cos\pi = -1$$

and

$$\sin\pi = 0$$

it follows that $e^{i\pi} = -1 + 0i$

which *yields Euler's identity* : $e^{i\pi} + 1 = 0$

Activities

- $e^0 = 1$
- $e^z \neq 0$
- $|e^z| = e^x \equiv e^{Rez}$
- $arg(e^z) = y + 2n\pi \equiv Imz \ , (n = 0, \mp 1, \mp 2, \dots)$

We know that $z' = x' + iy'$, *then* $\arg(z') = tan^{-1}(\frac{y'}{x'})$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\, cos(y) + ie^x\, sin(y) = x' + iy'$$

$$\arg(e^z) = \tan^{-1}\left(\frac{y'}{x'}\right) = \tan^{-1}\left(\frac{e^i \sin(y)}{e^x \cos(y)}\right) = \tan^{-1}(\tan(y)) = y + 2k\pi$$

$$\Rightarrow \quad arg(e^z) = y + 2\pi k$$

2022

- $\overline{e^z} = e^{-}$
- $\ln(e^z) = z + 2\pi i \quad , (n = 0, \mp 1, \mp 2, \dots)$
- $e^{\log z} = z$
- $\ln(z^{1/k}) = \frac{1}{k} \ln z \quad , \quad (k = \mp 1, \mp 2, \dots)$

Homework

1. Expand the follwing function

(a) $e^{-z} \quad ; \quad z = 0$

(b) $\cos z \quad ; \quad z = \pi/2$

(c) $\frac{1}{1+z} \quad ; \quad z = 1$

(d) $\ln z \quad ; \quad z = 2$

(e) $z\, e^{2z} \quad ; \quad z = -1$

2.Show that : $\sin z = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots \quad , \ |z| < \infty$

3.Show that : $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \quad , \ |z| < 1$

4.Show that : $\sec z = 1 + \frac{z^2}{2} + \frac{5}{24} + \cdots \quad , \ |z| < \pi/2$

5.Show that : $\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots \quad , \ 0 < |z| < \pi$

6. Expand : $\tan^{-1}(iz) =$ ... ...

Laurent's Series

The method of *Laurent series expansions* is an important tool in complex analysis. Where a *Taylor series* can only be used to describe the *analytic part* of a function, Laurent series allows us to *work around the singularities* of a complex function. To ***do this***, we need to determine ***the singularities*** of the function and can then ***construct several concentric rings*** *with the same center* z_0 based on those singularities and obtain a unique Laurent series of z-z_0 inside each ring where the ***function*** is *analytic*. ***In other words***,

If a function fails to be analytic at a point z_0 , one can not apply *Taylor's* theorem **at that point**.Unlike the Taylor series which express $f(z)$ as a series of terms with *non-negative powers* of z , a ***Laurent series*** includes terms with *negative powers*.

Theorem

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at z_0 and let C denote any poisitive orientated simple closed contour around z_0 and lying in that domain . Then,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

2022

تمثل C_n محيط الدائرة (الخارجية او الداخلية)التي نصف قطرها R_1 و R_2 وكلاهما بالاتجاه الموجب وعكس عقرب الساعة وعلى التوالي .

ملاحظـة :

1. يسمى الجزء $c_0 + c_1(z - z_0) + +c_2(z - z_0)^2 + \cdots$ بالجزء التحليلي من متسلسلة لورنت والجزء الثاني $\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$ بالجزء الاساسي .
2. اذا كان الجزء الاساس (الرئيسي) من متسلسلة لورنت (صفرا) فان متسلسلة لورنت تصبح " متسلسلة تايلر ".

Example 6 If $|z| > 1$, expand $f(z) = \frac{1}{1-z}$ using Laurent series

Ans. $\frac{1}{1-z} = -\sum_{n=1}^{\infty} \frac{1}{n}$, $|z| > 1$

$$f(z) = \frac{1}{1-z} = \frac{1}{z\left(\frac{1}{z}-1\right)} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\left(\frac{1}{z}\right)\left(\frac{1}{\left(1-\frac{1}{z}\right)}\right)$$

$$\frac{1}{1-z} = -\left(\frac{1}{z}\right)[1 + \frac{1}{z} + \frac{1}{z^2} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots.$$

$$\frac{1}{1-z} = -[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots.]$$

Here $f(z) = \frac{1}{1-z}$ is ana;lytic **everywhere** apart from the **singularity** at $z = \blacksquare$.

Above are the expansions of f in the regions inside and outside the circle of radius 1, centered on $z = 0$, where $|z| < 1$ is the region *inside* the circle and $|z| > 1$ is the region *outside* the circle.

Example 7 Expand $\frac{e^{2z}}{(z-1)^3}$; $z = 1$

Solution

Let , $u = z - 1 \rightarrow z = 1 + u$, $2z = 2(1 + u)$

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3} \times e^{2u} = \frac{e^2}{u^3}[1 + 2u + \frac{(2u)^2}{2!} + \cdots.$$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \cdots.$$

حيث ان $z = 1$ هو قطب (pole) من الرتبة الثالثة .

2022

Example 8 Expand $\frac{1}{z^2(z-3)^2}$; $z = 3$

Solution

Let , $u = z - 3 \rightarrow z = u + 3$, $z = u + 3$

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^2 + \cdots\}$$

$$\therefore \quad \frac{1}{z^2(z-3)^2} = \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4u}{24^3} + \cdots$$

$$\frac{1}{z^2(z-3)^2} = \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4u}{24^3} + \cdots$$

حيث ان $z = 3$ هو قطب (pole) من الرتبة الثانية .

Example 9 Let $f(z) = \frac{1}{2+z}$ .Determine the Laurent series around $z = 1$

Solution

Obviously, we have a simple pole at $z = -2$. Hence, we are dealing with a radius of **3** and want to find the Laurent series for both $|z - 1| < 3$ *and* $|z - 1| > 3$. The Laurent series will reduce to a Taylor series inside $|z - 1| < 3$ where $f(z)$ is analytic.

For $|z - 1| < 3$, we refer to the well-known geometric series. We begin by trying to create a $(z - 1)$ term in the denominator.

$$f(z) = \frac{1}{2+z} = \frac{1}{2+z-1+1} = \frac{1}{3+(z-1)} = (\frac{1}{3})\frac{1}{1-\left(\frac{-(z-1)}{3}\right)}$$

Since $\left|\frac{-(z-1)}{3}\right| < 1$, we can now represent the function as a series:

$$\Rightarrow \quad f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{-(z-1)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}} \quad for \quad |z-1| < 3$$

Which is just a Taylor series as the function is analytic inside the region. For $|z - 1| > 3$, we can use $\frac{3}{|z-1|}$ and follow our previous work to obtain:

$$f(z) = \frac{1}{3+(z-1)} = \frac{1}{z-1}\frac{1}{1-(\frac{-3}{z-1})} = \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(z-1)^n}$$

In this case ,we have obtained the Laurent expansion.The generalized residue for the outer ring $|z-1|>3$ is the coefficient of $\frac{1}{z-1}$, that is $b_1 = 1$.

Example 10 Expand $f(z) = \frac{1}{(z+1)(z+3)}$ as Laurent series when:

(a) $1 < |z| < 3$

(b) $|z| < 3$

(c) $0 < |z+1| < 2$

(d) $|z| < 1$

Solution

Using partial fractions

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$1 = Az + 3A + Bz + B$$

$$A + B = 0$$

$$3A + B = 1$$

$$A = -B$$

$$3A - A = 1 \rightarrow \quad \boldsymbol{A = 1/2}$$

$$\boldsymbol{B = -1/2}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

(a) $|z| > 1$, *then*

$$\frac{1}{2}\left(\frac{1}{z+1}\right) = \frac{1}{2z(1+\frac{1}{z})} = \frac{1}{2z}[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

(b) $|z| < 3$, *then*

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6(1+\frac{z}{3})} = \frac{1}{6}[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots]$$

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

وعليه فان مفكوك لورنت صحيح لكل من $|z| > 1$, $|z| < 3$ اي ان $|z| > 3$ ينتج بالطرح وهو :

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

2022

(c) $0 < |z+1| < 2$

Let , $u = 1 + z$, then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

$$0 < |z+1| < 2 \quad or, \quad |u| < 2 \ , \quad u \neq 0$$

(d) If $|z| < 1$ *, then*

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots$$

$$= \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

اذا كان $|z| < 3$ ، نجد من الفرع (a)

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

وبالتالي فان مفكوك لورنت المطلوب والصحيح لكل من $|z| < 1$ ، $|z| < 3$ ، اي ان $|z| < 1$ ينتج بالطرح :

$$= \frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

وهي متسلسلة تايلر .

Example 11!! Determine the Laurent series for $f(z) = \frac{1}{(z+5)}$ that are valid in the regions***:***

(a) $|z| < 5$

(b) $|z| > 5$

NOTE *:*

$$f(z) = \frac{1}{5(1+\frac{z}{5})} = \cdots. \quad , \quad f(z) = \frac{1}{z\left(1+\frac{5}{z}\right)} = \cdots.$$

Example 12!! Determine the Laurent series for $f(z) = \frac{1}{(z-i)^2}$ ***;*** $z_0 = i$

2022

Homework

1. Expand the follwing function

(a) $\frac{1}{\sqrt{1+z^3}}$; $z = 1$

(b)) $\frac{1}{\sqrt{1+z^3}}$; $z = 0$

$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{}$

$(2)(4)(6)^{z^9}$

(c)$sin^{-1}z =$

$-\frac{3}{3} + \frac{(1)(3)}{(2)(4)}\frac{z^5}{5} + \frac{(1)(3)(5)}{(2)(4)(6)}\frac{z^7}{7}$; $|z| < 1$

(d) $f(z) = \ln(3 - iz), f(0) = ln3$

(e)) $f(z) = \frac{1}{z-3}$ *as Laurent series for*

(a)$|z| < 3$

(b) $|z| > 3$

Ans.

(a)$\frac{-1}{z} - \frac{z}{9} - \frac{z^2}{9} - \frac{z^3}{81} + ..$

(b) $\frac{-1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \ldots.$

2.Expand :

$\frac{z}{(z-1)(2-z)}$ as Laurent series for ,

(a) $|z| < 1$, $(b)1 < |z| < 2$, (c) $|z| > 2$, $(d)|z - 1| < 1$, $(e)0 < |z - 2| < 1$

Ans.

(a) $\frac{-1}{2}z - \frac{3z^2}{4} - \frac{7z}{8} - \frac{15z^3}{16} - \ldots.$

(b)$\frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \ldots.$

(c) $\frac{-1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} + \cdots.$

(d)$\frac{-1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} + \ldots.$

(e) $1 - \frac{2}{(z-2)} - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} + \ldots.$

3.Expand : $f(z) = \frac{1}{z(z-2)}$; $0 < |z| < 2$, $|z| > 2$

4.Expand : $f(z) = \frac{z}{1+z^2}$; , $|z - 3| > 2$

5.Expand : $f(z) = \frac{1}{(z-2)^2}$; $|z| < 2$, $|z| > 2$

6.Expand the following functions as Laurent series around $z = 0$

(a) $f(z) = \frac{1-cosz}{z}$, [*Ans.* $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} \ldots$

2022

(b) $f(z) = \frac{e^{z^2}}{z^3}$, [*Ans.* $\frac{1}{z^3} - \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \frac{z^5}{4!} + \frac{z^7}{7!} + \cdots$

(c) $f(z) = zsinh\sqrt{z}$, (d) $f(z) = ze^z$

7. State the singular points for the functions :

$(a)\ \frac{1}{2(sinz-1)^2}$, $(b)\ \frac{z}{(e^{1/z}-1)^2}$, $(c)\cos(z^2 - z^{-2})$, $(d)\ \frac{z}{(e^{1/z}-1)}$

$(e)\ tan^{-1}(z^2 + 2z + 2)$, $(f)\ (\frac{z}{(e^z-1)}$

Complex Integrals

المحاضرة 12

✓ التكاملات الخطية العقدية (Complex line integral)

Liouvalli theorem (a)

Morera's theorem (b)

✓ تكاملات كوشي (Cauchy's integral)

From " Calculus " , the integration say for example

- $\int_0^1 f(x)dx$, dx with be varied from $x_1 = 0$, $x_2 = 1$.
- $\int_2^3 f(y)dy$, dy with be varied from $y_1 = 2$, $y_2 = 3$.
- $\int_{1,2}^{3,4} (3xy + y^2)dx + (2x^2y - yx + x^2)dy$,

هنا التكامل يتضمن دوال معتمدة على x و y . هذه جميعا " تكاملات خطية " ولابد ان توجد علاقة تربط y مع x ، على سبيل المثال :

1. $y = 2x$ *striaght line equation*
2. $y = x^2$ *parabola equation*

ففي الحالة الاولى $y = 2x$ يتم اعطاء قيم y مباشرة او قيمة حيث نعوض عن او بما يساويها فتتحول المعادلة بدلالة بعد واحد ، أما :

$$dy = 2dx$$

$$dy = 2xdx$$

الحالة الثانية ، يتم تحديد " نقطتان " في السوال ضمن المعطيات ومن خلال القيمتين يتم ايجاد قيمة أما y أو x عن طريقة معادلة الخط المستقيم وابجاد الميل وكالاتي :

Along the striagt line joining the points $(1,2) \rightarrow (3,4)$

بحيث يكون C هو المستقيم الواصل بين النقطتين $(1,2) \rightarrow (3,4)$ ، أي

2022

$$slope = m = \frac{y_2 - y_1}{\quad} \qquad \frac{}{3-1} = \frac{2}{2} = 1$$

$$m = 1$$

من معادلة (ميل + نقطة) ،

$$y_2 - y_1 = m(x_2 - x_1)$$

$$y - 2 = (1)(x - 1)$$

$$\therefore \quad y = x + 1 \ \rightarrow dy = dx$$

Complex line integral التكاملات الخطية العقديــة

Previous lecture ,

Complex function $\quad w = f(z) = U(x,y) + \boldsymbol{i}\, V(x,y)$

Since , $\quad z = x + iy$

Let , $\qquad \Delta z = \Delta x + i\Delta y \quad$,or $\quad dz = dx + idy$

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$\therefore \quad \int_C f(z)dz = \underbrace{\int_C udx - \int_C vdy}_{\text{Real part of line integral}} + \boldsymbol{i}\underbrace{\int_C udy + \boldsymbol{i}\int_C vdx}_{\text{Imaginary part of line integral}}$$

Example 1 Evaluate the integral $\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz \ \ where \ \ y = 2x^2$

Solution

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz = \int_{(1,1)}^{(2,8)} (x^2 - iy^2)(dx + idy)$$

2022

$$= \int_{(1,1)}^{(2,8)} x^2dx + \int_{(1,1)}^{(2,8)} y^2dy + \boldsymbol{i} \int_{(1,1)}^{(2,8)} x^2dy - \boldsymbol{i} \int_{(1,1)}^{(2,8)} y^2dx$$

$$\because \quad y = 2x^2 \quad \rightarrow dy = 4xdx$$

$$y^2 = 4x^4$$

Substitute in integration equation

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz = \int_1^2 x^2dx + \int_1^2 16x^5dx + \boldsymbol{i} \int_1^2 4x^3dx - \boldsymbol{i} \int_1^2 4x^4dx$$

$$= \frac{x^3}{3}\Big|_1^2 + \frac{16x^6}{6}\Big|_1^2 + \boldsymbol{i}x^4|^2 - \boldsymbol{i}\frac{4x^5}{3}\Big|_1^2$$

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz = \left(\frac{8}{3} - \frac{1}{3}\right) + \left(\frac{1024}{6} - \frac{16}{6}\right) + i\left(16 - 10 - i\left(\frac{128}{5} - \frac{4}{5}\right)\right)$$

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz = \frac{511}{3} + \boldsymbol{i}\frac{49}{5}$$

ملاحظة : يمكن التعبير عن المتغيرين x و y كدوال بمتغير واحد لنقل t والذي يسمى " وسيطا " بين المتغير x و y كما موضح في المثال التالي .

Example 2 Evaluate the integral $\int_C z^2dz$ where C is striaght line joining the points : $(0,0) \rightarrow (2,1)$,where C: $Z(t) = x(t) + iy(t)$

Solution

$$y - y_1 = m(x - x_1) \qquad \text{"straight line equation"}$$

$$slope = m = \frac{y_2 - y_1}{\quad} \qquad \frac{}{2 - 0} = \frac{1}{2}$$

$$m = \frac{1}{2}$$

$$y - 0 = \frac{1}{2}(x - 0)$$

2022

$$\therefore \quad y = \frac{1}{2}x \rightarrow x = 2y$$

ومن الواضح ان x ، y هما دالتين للمتغير t (حسب ماورد في السوال) وعليه يمكن ان نفترض :

Let , $y = t \quad \rightarrow x = 2t$ then $0 \le t \le 1$

$$\therefore \quad C \quad : \quad z(t) = 2t + it \qquad 0 \le t \le 1$$

$$dz = 2dt + idt$$

Or ,

$$dz = (2+i)dt$$

Now ,

$$\int_C z^2 dz = \int_0^1 (2t+it)^2\,(2+i)dx$$

$$= (2+i)\int_0^1 (4t^2 + 4t^2 i - t^2)dt$$

$$= (2+i)\int_0^1 (3t^2 + i4t^2)dt$$

$$\int_C z^2 dz = (2+i)\left[t^3 + \frac{4}{3}it^3\right]\Big|_0^1 = (2+i)(1+i\frac{4}{3})$$

$$\int_C z^2 dz = \frac{2}{3} + \boldsymbol{i}\frac{11}{3}$$

Activity *:*

If we choose $y = \frac{t}{2}$, $x = t$ … continue

Example 3 Evaluate the integral $\int_C z^2 dz$ where path equation C is $Z(t) = t + it^2 \qquad 0 \le t \le 1$

Solution

$$Z(t) = t + it^2$$

$$\therefore \quad dz = dt + \boldsymbol{i}\,2tdt$$

2022

$$\int_C z^2 dz = \int_0^1 (t+it^2)(1+2it)dt \quad , \quad 0 \le t \le 1$$

$$= \frac{(t+it^2)^3}{3} \mathrm{I}^1 = \frac{1}{3}(1+i)^3 = \frac{-2}{3} + \frac{2}{3}\boldsymbol{i}$$

$$\int_C z^2 dz = \frac{-2}{3} + \frac{2}{3}\boldsymbol{i}$$

نظرية كوشي –كورسات Cauchy-Goursat theorem

The Cauchy–Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint_C f(z)\, dz = 0.$$

لتكن $f(z)$ دالة تحليلية في منطقة ما $\mathcal{R}$ وكذلك على حدودها C او اذا كانت $f(z)$ دالة تحليلية في كل نقطة على وداخل منحني مغلق بسيط C فان :

$$\oint_C f(z)dz = 0$$

Example 4 If C is a circle $|z| = 1$, prove that $\oint_C \frac{z^2}{z-3} dz = 0$

Solution

$$f(z) = \frac{z^2}{z-3} \quad , \quad z_0 = 3$$

z_0 تمثل نقطة او قطب (شاذ) ، اي ان الدالة تحليلية لجميع النقاط في المستوى العقدي عدا النقطة $z_0 = 3$, لانها تقع (خارج الدائرة $|z| = 1$) وعليه فان قيمة التكامل تساوي صفر .

ملاحظـة : الدالة العقدية هي $\frac{z^2}{z-3}$ كما موضح في السؤال .

Example 5 If C is $|z| = 2$, evaluate $\int \frac{1}{z-3} dz$

Solution

$\int \frac{1}{z-3} dz$, not define at $z = 3$ not in $C : |z| = 2$, hence

$\int \frac{1}{z-3} dz = 0$

Example 6 Find the integral $\oint_C \frac{dz}{(z-1)^2(z^2+4)}$ where C is $3 \leq |z| \leq 5$,

Solution

$f(z) = \frac{1}{(z-1)^2(z^2+4)}$

$$(z-1)^2 = 0 \quad \rightarrow \boldsymbol{z = 1}$$

$$z^2 + 4 = 0 \quad \rightarrow \quad z^2 = -4 \quad \rightarrow \boldsymbol{z = \mp 2i}$$

اذن الدالة تحليلية في جميع نقاط المستوى العقدي عدا النقاط

$$\boldsymbol{z = 1 \ , \ z = +2i \ , z = -2i}$$

وجميعها **خارج المنطقة** وعليه فان **قيمة التكامل تساوي صفر** .

نظرية موريرا (Morera's Theorem)

وهي عكس نظرية كوشي – ريمان وتنص على الاتي :

أذا كانت $f(z)$ دالة مستمرة في المنطقة $\mathcal{R}$ وان $\oint f(z)dz = 0$ حيث C هو منحني بسيط مغلق عندئذ تكون $f(z)$ دالة تحليلية في المنطقة $\mathcal{R}$.

نظرية ليوفيل (Louvelli's Theorem)

اذا كانت $f(z)$ دالة كلية (Entire function) ومحددة لكل قيم z في المستوى العقدي عندئذ تكون $f(z)$ دالة ثابتة او بعبارة اخرى ، افرض ان $f(z)$ دالة تحليلية محددة اي ان $|f(z)| < M)$ لثابت ما M لجميع قيم z في المستوى العقدي الشامل ، ينتج ذلك ان $f(z)$ يجب ان تكون ثابتة .

صيغ كوشي التكاملية (Cauchy's integral formulas)

يمكن تقسيمها الى قسمين هما :

(1) ذات القطب البسيط (Simple pole)

(2) ذات القطب المتعدد (Multipole pole)

2022

اذا كانت $f(z)$ تحليلية على المنحني البسيط C وداخلها وان z_0 اي نقطة داخل C والحركة عكس عقرب الساعة (تكون موجبة) فان :

$$(1)\oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z)$$

$$(2)\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^n(z)$$

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 0,1,2,\dots.)$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz \qquad (n = 1,2,\dots.)$$

Example 7 Evalute $\oint_C \frac{e^z}{(z-1)}dz\,, \quad c\ :\ |z| = 2$

Solution

$$z_0 = 1$$

$$\oint_C \frac{e^z}{(z-1)}dz = 2\pi i f(z_0)$$

$$f(z) = e^z \ \rightarrow\ f(z_0) = e^{z_0} = e^1$$

$$\oint_C \frac{e^z}{(z-1)}dz = 2\pi\, i e^1$$

Example 8 Evalute $\oint_C \frac{cosz}{(z+\pi)}dz\,, \quad c\ :\ |z| = 3$

$$f(z) = cosz\,, \quad z_0 = -\pi\ , f(z_0) = \cos(-\pi)$$

$$\oint_C \frac{cosz}{(z+\pi)}dz = 2\pi i cos(-\pi) = -2\pi i$$

Example 9 Evalute $\oint_C \frac{e^z}{z^2}dz\,, \quad c\ :\ |z| = 2$

$$f(z) = e^z\,, \quad z_0 = 0\ , f(z_0) = f(z) = e^{z_0} = e^0 = 1$$

$$\oint_C \frac{e^z}{z^2}dz = 2\pi i(e^0) = 2\pi i$$

Example 10 Evalute $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz\,, \quad if\ \ c\ \ is:\ (a)\ |z| = 3\ ,(b)\ \ |z| = 1$

$$(a)\ |z| = 3$$

$$z_0 = 2\ , f(z) = e^z\,, \qquad f(z_0) = e^2$$

$\therefore \oint_C \frac{e^z}{z-2} dz = 2\pi i(e^2)$

$$\therefore \frac{1}{2\pi i} \times 2\pi i(e^2) = e^2$$

$(a)\ |z| = 2$

وحسب نظرية كوشي كورسات $z_0 = 2$ ، فان

$$\therefore \oint_C \frac{e^z}{z-2} dz = 0$$

لان الدالة e^z تحليلية في جميع المناطق للمستوى العقدي ماعدا $z_0 = 2$ والتي تقع خارج المنحني $|z| = 1$.

Example 11 Evalute $\oint_C \frac{sin^2 z}{z-\pi/6} dz\,,\quad if\ \ c\ \ is: _|z| = 1$

Solution

$$z_0 = \pi/6 \quad , f(z) = sin^2 z$$

$$\therefore f(z) = sin^2(\pi/6) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\oint_C \frac{sin^2 z}{\sin(-\pi/6)} dz = 2\pi i\left(\frac{1}{64}\right) = \frac{\pi}{32} i$$

Example 12 Evalute $\oint_C \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz\,,\quad if\ \ c\ \ is: _|z| = 3$

Solution

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\oint \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz = \oint \frac{sin\pi z^2 + cos\pi z^2}{(z-1)} dz - \oint \frac{sin\pi z^2 + cos\pi z^2}{(z-2)} dz$$

$$f(z) = sin\pi z^2 + cos\pi z^2\,,\quad z_0 = 2\,,\ z_0 = 1$$

$$f(z_0)_2 = sin\pi(2)^2 + cos\pi(2)^2 = 1$$
$$f(z_0)_1 = sin\pi(1)^2 + cos\pi(1)^2 = -1$$

2022

Then,

$$\oint \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i$$

لان $z_0 = 1$ ، $z_0 = 2$ داخل المنحني C .

Example 13 Evalute $\oint_C \frac{cosz}{(z-\pi)^5} dz$

Solution

Since

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$f^{(4)}(z_0) = \underset{1}{-sinz} \rightarrow \underset{2}{-cosz} \rightarrow \underset{3}{sinz} \rightarrow \underset{4}{cosz}$$

$$z_0 = \pi \quad , n = 4$$

$$\oint_C \frac{cosz}{(z-\pi)^5} dz = \frac{2\pi i}{4*3*2*1} cos\pi = -\frac{\pi}{12} i$$

Example 13 Evalute $\oint_C \frac{e^z}{(z+1)^4} dz$ where c is a circle $|z| = 3$

Solution

$$f(z) = e^{2z} \quad , z_0 = -1 \quad , \quad n = 3$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$f^{(3)}(z) = 8e^{2z} \quad , f^{(3)}(z_0) = f^{(3)}(-1) = 8e^{-2}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3*2*1} \times 8e^{-2} = \frac{8\pi i}{3} e^{-2}$$

2022

Example 14 Evalute $\oint_C \frac{e^{iz}}{z^3} dz$ where c is a circle $|z| = 2$

Solution

$$z_0 = 0 \quad , \quad n = 3 \quad , \quad f(z) = e^{iz}$$

$$f'(z) = ie^{iz} \quad , \qquad f''(z) = -e^{iz} \quad , \qquad f''(z_0) = f''(0) = 1$$

$$\oint_C \frac{e^{iz}}{z^3} dz = \frac{2\pi i}{2 * 1}(-1) = -\pi i$$

$$\oint_C \frac{e^{iz}}{z^3} dz = -\pi i$$

2022

Complex Integration with Residue

المحاضرة 13

✓ التكامل العقدي باستخدام الرواسب (البواقي) (Residue)

نفرض ان $f(z)$ دالة تحليلية على الدائرة C وداخلها وان z_0 نقطة شاذة معزولة للدالة $f(z)$ ، فان مفكوك متسلسلة لورنت للدالة $f(z)$ هو :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (1)$$

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots . + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad , n = 0, \mp 1, \mp 2, \ldots \qquad (2)$$

عندما كانت $n = -1$ فان المعادلة (2) ،

$$a_{-1} = \frac{1}{2\pi i} \oint f(z)$$

يسمى (a_{-1}) راسب الدالة $f(z)$ عند النقطة z_0 ويرمز له $Res(f, z)$ وعليه فان له $Res(f, z) = a_{-1}$ ، لذلك يكون

$$\oint f(z)$$

كذلك اذا كان للدالة $f(z)$ قطب من الرتبة n عند النقطة z_0 فان :

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{f(z)(z - z_0)^n\}$$

2022

Example 1 Evaluate the integral $\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz$ where C is a circle $|z-3|=1$ in positive direction.

Solution

القطب من الرتبة الثالثة $z_0=\pi$

$$Res(f,z_0)=\frac{1}{(n-1)!}lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\left\{(z-\pi)^3\frac{e^{iz}-sinz}{(z-\pi)^3}\right\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\{e^{iz}-sinz\}=\frac{1}{2}$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=2\pi i\,Res(f,\pi)=2\pi i*\frac{1}{2}=\pi i$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=\pi i$$

Example 2 Use the residue theorem to find $\oint_C \frac{dz}{(z-1)(z+1)}$ where C: $|z|=3$.

Solution

$$z_0=1\quad ,z_0=-1$$

تمثلان قطبين بسيطين للدالة $f(z)$ ويقعان داخل الدائرة $|z|=3$.

$$f(z)=\frac{1}{(z-1)(z+1)}$$

1. $z_0=1$

$$Res(f,1)=lim_{z\to1}\left\{(z-1)\frac{1}{(z-1)(z+1)}\right\}=\frac{1}{2}$$

2. $z_0=-1$

$$Res(f,-1)=lim_{z\to-1}\left\{(z+1)\frac{1}{(z+1)(z+1)}\right\}=\frac{-1}{2}$$

$$\oint_C \frac{dz}{(z-1)(z+1)}=2\pi i[Res(f,1)+Res(f,-1)]$$

2022

$$\oint_C \frac{dz}{(z-1)(z+1)} = 2\pi i\left[\frac{1}{2} - -\frac{1}{2}\right] = 0$$

$$\oint_C \frac{dz}{(z-1)(z+1)} = 0$$

Example 3 Find the residue of function $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$.

Solution

$$z_0 = -1 \quad , \quad z_0 = \mp 2i \ , \qquad n = 2$$

$$z_0 = -1 \quad , \quad z_0 = \mp 2i \ , \qquad n = 2$$

1. Residue at $z_0 = -1$

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f, -1) = \frac{1}{1!} lim_{z\to -1} \frac{d}{dz}\left\{(z+1)^2 \times \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$$

$$es(f, -1) = lim_{z\to -1}\left\{\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\right\} = -\frac{14}{25}$$

2. Residue at $z_0 = 2i$

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f, -1) = \frac{1}{1!} lim_{z\to 2i} \frac{d}{dz}\left\{(z-2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f, -1) = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

3. Residue at $z_0 = -2i$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to -2i} \frac{d}{dz}\left\{(z+2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f,-1) = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

Example 4 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz$ where C is a circle $|z| = 3$.

Solution

$$f(z) = \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$z_0 = 0 \quad , \quad z_0 = -1 \mp i \ , \qquad n = 2$$

قطب من الرتبة الثانية

1. $$Res(f,-1) = \frac{1}{1!} lim_{z\to 0} \frac{d}{dz}\left\{z^2 \times \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= lim_{z\to 0}\left\{\frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{z^2(z^2+2z+2)}\right\}$$

$$Res(f,-1) = \frac{t-1}{2}$$

2. Residue at $z_0 = -1+i$

$$Res(f,-1+i) = lim_{z\to -1+i}\left\{\{z-(-1+i)\} \times \frac{e^{zt}}{z^2\{z-(-1+i)\}\{z-(-1-i)\}}\right\}$$

$$Res(f,-1+i) = \frac{e^{(-1+i)t}}{(-1+i)^2} \times \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

3. Residue at $z_0 = -1-i$

$$Res(f,-1+i) = lim_{z\to -1-i}\left\{\{z-(-1-i)\} \times \frac{e^{zt}}{z^2\{z-(-1-i)\}\{z-(-1+i)\}}\right\}$$

2022

$$Res(f,-1+i)=\frac{e^{(-1-i)t}}{(-2i)(-1-i)^2}=\frac{e^{(-1-i)t}}{4}$$

مجموع الرواسب ،

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i$$

$$=2\pi i\left[\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right]$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i\left[\frac{t-1}{2}+\frac{e^{-t}}{2}cost\right]$$

اي ان ،

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=\frac{1}{2}(t-1)+\frac{1}{2}e^{-t}cost$$